LEARNING FROM OTHERS

Paul Andrews and Judy Sayers share a European perspective on teaching linear equations in Finland, Flanders, and Hungary.

Setting the scene

We begin this article with an assertion that it is through observation, in its broadest sense, and opportunities for participation, that we assimilate ourselves into the culture in which we are raised. As children, we learn from watching other, usually older, children playing. Watching our parents and comparing their behaviour towards us with that of our friends’ parents is how we acquire our perspectives, at least initially, on gender roles and parenting, both of which we know, from sociological and anthropological research, are culturally located. As children, our home lives induct us, usually unconsciously, into the norms, values and expectations of the social class in which we are located. There may be tensions and challenges to those values as we meet others with different values and expectations, but essentially our families locate us within a particular world view from which it is often difficult to escape. Participating in the routines of the educational institutions we attend, particularly when young, locates us in another set of values and we are all aware, as teachers, of the tensions that can arise when the values a child brings to school are different from those the system wishes to impart. In short, the ways we come to understand our roles as teachers and, by implication, agents of cultural reproduction are informed greatly by our experiences, frequently vicarious, as observers and participants in the system we come to represent.

However, observation, without circumspection, and participation without reflection, may not always be the most productive way of learning. Observation and participation, as promulgated initially by the Conservative administrations of the 1990s and currently in the rhetoric of the 2010s, are unlikely to improve the quality of teaching in English schools because they are rooted in some sense of what has always been. Observation and participation, if they are to be effective, require an awareness of alternatives sufficient to mount warranted critiques and an understanding of what is possible beyond what is current. This is why it is unlikely that we could become effective, or even reflective, teachers solely by observing others working in the same cultural tradition. The uncritical introspection embedded in the Thatcherite legacy tells us that attempts to reform practice that fail to look outside the system are doomed to replicate rather than change that which has always been.

Between us we have worked in teacher education for nearly thirty years. During that period, we have observed several introductions to linear equations which, with few exceptions, have tended to follow a similar format. The lesson starts with teachers rehearsing mental strategies for solving orally presented, problems of the form, I’m thinking of a number, I multiply it by three, I subtract four and get an answer of eleven. What was my number? Typically students solve the equation by means of a process of operation reversal derived from their understanding of arithmetic. Once trainees are satisfied that their students have acquired sufficient fluency with the procedure, a procedure essentially operationalising an implicit definition of equation, they move towards the main objective of the lesson; the teaching of analytic strategies for solving linear equations. The means by which this process is undertaken varies, although three approaches seem typical. We have seen trainees work discursively towards a formalisation of the undoing process described above. We have seen trainees encourage their students to construe an equation as a balance scale and work towards an articulation of the do-the-same-to-both-sides principle. Finally, we have seen trainees tell their students that solving equations...
entails learning that changing the side means changing the signs.

It is not our intention to comment on these approaches as all have both advocates and critics. It is our intention to reflect on the one characteristic common to all three approaches and to compare that commonality with what we have seen in other European countries. This interesting commonality is that trainees have always begun their formal treatment with arithmetic equations; equations with the unknown on one side that they know their students can solve mentally. Consequently, we have seen on disappointingly many occasions, trainees appearing genuinely surprised when their students ask, why are we doing this, you know we can do it already?

So, if we have only ever seen English trainees introduce linear equations through arithmetic equations, approaches that must have been sanctioned by their school-based mentors as part of their induction into English mathematics teaching, then what have we seen in European classrooms that is different from this tradition? Interestingly, the didactic sequences we have seen in Finland, Flanders, and Hungary have also started with the sorts of mental activities observed in English classrooms. However, the major difference, it seems to us, is that our European colleagues use these mental activities to serve a simple but very important purpose – to lull their students into a false sense of security before they begin their formal treatments of the topic. They do this because, unlike the English trainees we have observed, they all started their formal treatments with algebraic equations, equations with the unknown on both sides. Thus, they began with equations they know their students can’t solve in their heads. However, the manner in which these algebraic equations are introduced varies, and in the following we describe three sequences of lessons taught by a teacher from each of Finland, Hungary, and Flanders.

The Finnish teacher, Sami

Sami introduced his students to the notion of an equation by writing on the board that an equation is two expressions connected by an equals sign. He operationalised his definition by asking his students to evaluate a number of examples against it. Sami ended this lesson by presenting a range of arithmetic equations, including some with fractional coefficients, for solving by a process of undoing.

The second lesson began with Sami writing on the board $5x + 3 = 2x - 8$ and inviting solutions. He waited for several seconds before announcing, with something of a mischievous smile, “it’s not that easy anymore, is it?” Next, he talked of balance scales and how the same operation applied to both sides would sustain the balance. He wrote nothing, but exploited his outstretched arms to demonstrate the effect of different actions on the scales while commenting that “an equation is like scales... in principle, if you have it in balance, the equation is true”. Returning to the equation, he asked what could be subtracted from both sides of the equation. Someone suggested $x$ and Sami, without comment or question, wrote $4x + 3 = x - 8$. Another volunteer suggested subtracting $2x$, at which point Sami wrote, with little student input:

$$
\begin{align*}
5x + 3 &= 2x - 8 \\
3x + 3 &= -8 \\
3x &= -11
\end{align*}
$$

After some student uncertainty with regard to the next step, Sami, having asserted that they should divide by three as division is the opposite of
multiplication, led the class to the solution \( x = -\frac{11}{3} \). Lastly, individual seatwork, involving similar algebraic equations saw the lesson to an end.

After checking the results of the previous lesson’s homework, Sami began the third lesson by solving publicly \( 3(x + 2) - 5 = -4(x - 2) \). This allowed him to demonstrate the need to remove brackets, before he set another exercise that lasted the remainder of the lesson. In similar fashion the next lesson included an invocation to change the side change the sign, while the last lesson, in addition to another exercise, saw students presented with one equation satisfied by all values of \( x \) and one that led to a contradiction, implying it had no solution.

**The Flemish teacher, Pauline**

Pauline framed the whole topic by posing a problem concerning the children, Bart, Lisa and Maggie, of the cartoon family, the Simpsons, and their mother, Marge. They were asked, if Marge is 33, Bart, 10, Lisa, 7 and Maggie, 0, in how many years would the sum of the children’s ages equal their mother’s?

Students were invited to construct a table showing the sum of the children’s ages with each annual increment. From the table students could see that after eight years the two totals would be the same. Pauline then encouraged the students to think about how they could use \( x \) to describe the unknown number of years, from which the equation \( x + 33 = 3x + 17 \) emerged. Students were then asked to draw the two graphs, \( y = x + 33 \) and \( y = 3x + 17 \) to see where they crossed. Finally, with regard to this episode, Pauline informed the class that soon they would have ways of solving such problems more rapidly before asking them to solve mentally four arithmetic equations. This was followed by her sumarising the relationship between each of her four exemplars and their respective formalisations. That is, over four rows, she wrote:

\[
\begin{align*}
   a &= b & \Rightarrow a + c &= b + c; \\
   a - c &= b - c; \\
   a.c &= b.c \\
   a/c &= b/c.
\end{align*}
\]

The first lesson ended with her posing a cross-word like task whereby students were to solve a number of arithmetic equations similar to those above. Midway through her second lesson, having checked the answers to the previous exercise, Pauline warned her students that a harder example was about to follow and wrote \( 6(x - 5) - 8 = x - 3 \) on the board. This was the basis for her formal treatment in which the algebra, including actions, was written on the left side of the board and justificatory annotations on the right. Throughout the process, which lasted more than twenty minutes and invoked obliquely the notion of a balance, Pauline questioned continuously. What follows represents a fragment of what was written.

\[
6(x - 5) - 8 = x - 3 \quad \text{(1) Eliminate brackets}
\]

At this point, Pauline elicited from her students that the principle of distributivity justified her next step.

\[
6x - 30 - 8 = x - 3 \quad \text{(2) Calculate if possible}
\]

\[
6x - 38 = x - 3
\]

Eventually, after obtaining a solution and discussing its uniqueness, Pauline undertook a check. Over the next three lessons Pauline presented a series of exercises, each of which was undertaken individually before answers were shared publicly. These exercises incorporated brackets and eventually included fractional coefficients. During the final lesson Pauline set a fifteen minute three item test comprising \( 5(p + 2) = 6p - 3; \) \( (14 - 2x) - (x + 12) = x - 2 \) and \( -\frac{3}{4}y = -\frac{2}{3} \).

**The Hungarian teacher, Emese**

Emese began her sequence with a discussion through which an equation was defined as comprising two expressions connected by an equals sign. She asserted, through her questioning, that equations may, or may not, contain variables or unknowns depending on circumstances and that they were always true, sometimes true, or never true. Finally she operationalised her definition through an exercise in which three open sentences, \( 5 - \Box = 8, 5 - \Box > 6 \) and \( \Box = 7 \), were solved in relation to the basic set \( -3 \leq \Box \leq 3 \).

Next she posed a series of oral problems like, *Kala is twice as old as her sister, the sum of their ages is 24, how old are they?* Each was solved individually before solutions were shared. Finally, on the first day, the class was split into four groups with each given a superficially different word problem for translating into an equation. One group’s problem was: *Some friends went on a trip. The first day they covered just 2km, the second day they covered \( \frac{2}{5} \) of the remaining journey. If they covered 6km on the second day, how long was their journey?* After several minutes a group representative explained how its equation had been derived and wrote \( 0.2(x - 2) = 6 \) on the board. Lastly, a volunteer, exploiting a *thinking backwards* strategy, obtained a solution of 32, which Emese checked against the text of each problem.
Emese began her second lesson with a word problem: *On two consecutive days the same weight of potatoes was delivered to the school’s kitchen. On the first day 3 large bags and 2 bags of 10kg were delivered. On the second day 2 large bags and 7 bags of 10kg were delivered. If the weight of each large bag was the same, what weight of potatoes was in the large bag?*

Soon a volunteer wrote $3x + 20 = 2x + 70$. Then, having established that intuitive strategies were now insufficient, Emese drew a picture of a scale balance with the various bags represented on both sides. Drawing on a student’s suggestion Emese erased two small bags from each side, leaving a representation of $3x = 2x + 50$. Next she was told to erase two large bags from each side to show one large bag balancing five small. Then, in response to her request, students volunteered sufficient for her to write alongside her drawings:

\[
\begin{align*}
3x + 20 &= 2x + 70 \\
3x &= 2x + 50 \\
x &= 50 \text{ kg}
\end{align*}
\]

Finally, Emese reminded her class of the importance of checking, and did so. Emese began her third lesson by presenting equations with the unknowns on both sides for individual and then public working – a strategy employed in all remaining lessons. She focused attention on the balance principle and alerted her students to the difficulty of representing negative terms on the scales. On several occasions, two solutions were shared simultaneously and compared for elegance and efficiency, while in every case, solutions were checked. During the fourth and fifth lessons the balance was less explicitly exploited and an expectation that students should work with brackets and both negative and fractional coefficients emerged. Interspersed throughout were word problems; a typical being, “*A stake is driven through a pond into the ground. If \( \frac{2}{3} \) of the stake’s length is in the ground, \( \frac{2}{3} \) in water and 2.8m above the water, how long is the stake?*” Lastly, several equations alerted students to the fact that not all equations yield solutions.

**Bringing it all together**

There were strong elements of similarity in the ways in which all three teachers presented equations to their students. All three offered definitions, either explicitly or implicitly, which were operationalised through problems and exercises in varying explicit ways. All three, having activated students’ knowledge and skills, provoked analytical methods by posing an algebraic equation that could not be solved by intuitive methods. All based their expositions on the balance although the extent to which this was sustained varied considerably. All three offered extensive opportunities for consolidation that incorporated expectations of students managing brackets and different forms of coefficient alongside particular additional insights.

So, returning to our concerns about a process of professional development focused on observation and, essentially, repetition, what can we infer? We could start by commenting that none of our case study teachers expected their students to attend to an exposition focused on an equation long since solved mentally. More importantly, perhaps, we note that the English perspective on linear equations seems substantially at odds with that of effective European teachers. Admittedly, one could find cause for criticism in all three of the sequences we have described. For example, having theatrically introduced the balance, Sami elected not to pursue it in any systematic way. Even as he worked through the solution discussed here he made only scant use of the metaphor. Pauline, having provoked the topic with the Simpsons’ problem never returned to the equation the class had derived in order to solve it by means of the analytical processes she had so painstakingly developed. Emese, as can be seen in the picture of her chalkboard, represented the scales in such a way that the effect of moments made impossible its representing any genuine equilibrium. Indeed, having shown this video on many occasions over the last few years, a small number of teacher colleagues have expressed such concern over this failing that all other elements of Emese’s presentation were dismissed as irrelevant.

There were also interesting differences with respect to the three teachers’ conceptualisations of equations. Emese and Sami categorised equations as part of an attempt to locate equation solving within a wider network of concepts than just learning solution strategies. Pauline, while focusing explicitly on such strategies, exploited her students’ existing web of concepts to justify each step of a solution. Thus, in different ways, all three teachers attempted to locate their students’ understanding of equations, and equation solving, within a wider conceptual framework. The examples and exercises exploited by Sami were located entirely within a world of mathematics. Pauline, while focusing explicitly on such strategies, exploited her students’ existing web of concepts to justify each step of a solution. Thus, in different ways, all three teachers attempted to locate their students’ understanding of equations, and equation solving, within a wider conceptual framework. The examples and exercises exploited by Sami were located entirely within a world of mathematics. Pauline, while focusing explicitly on such strategies, exploited her students’ existing web of concepts to justify each step of a solution. Thus, in different ways, all three teachers attempted to locate their students’ understanding of equations, and equation solving, within a wider conceptual framework. The examples and exercises exploited by Sami were located entirely within a world of mathematics.
both countries, they formed part of a didactic sequence of ideas in which the introduction of new difficulties was managed by the teachers concerned, leaving students with rare opportunities to engage with non-routine problems. This was much less the case with Emese, whose word problems provoked much individual and collective engagement. This latter observation alludes to culturally located perspectives on classroom norms. Emese engaged her students in much collective activity focused on students’ awareness and acquisition of mathematical thinking. Pauline had clear objectives that were explicitly addressed by means of extensive, but tightly focused, bouts of public questioning. Sami exploited extensive bouts of teacher telling, interspersed with exercises, from which students were expected to infer meaning.

Such practices, while of substantial interest to anyone looking for warranted alternatives to traditional English mathematics teaching, must be seen in the context in which they occur. There is evidence that Finnish mathematics teaching, despite curricular changes lasting more than thirty years, has remained largely unchanged for the majority of the twentieth century, with the teacher talking for much of the time in such predictable ways that both subjects and teachers could be interchanged and no one would notice the difference (Norris et al., 1996). Almost despairingly, Carlgren et al. (2006: 314) describe the Finnish comprehensive classroom as a “wasteland not only of intelligence but also of emotions”. Thus, while Sami shared many aspects of his approach with his Flemish and Hungarian colleagues, much of what he did seemed located in such a tradition from which, we conclude, little of didactic value may be learned. Pauline’s tradition is different. Informal conversations with various Flemish colleagues, working in the UK and Flanders, have confirmed our perception that Flemish mathematics teaching draws on the rigours of Bourbaki and the new mathematics of the 1960s and 70s, but is mediated by the humanising impact of Bourbaki and the new mathematics of the 1960s. Informal conversations with various Flemish colleagues, working in the UK and Flanders, have confirmed our perception that Flemish mathematics teaching draws on the rigours of Bourbaki and the new mathematics of the 1960s and 70s, but is mediated by the humanising impact of the Dutch realistic mathematics education (RME) movement. Thus, for example, the impact of RME could be seen in Pauline’s use of the Simpsons’ problem while the confident use vocabulary like distributivity, reflected a clear Bourbaki perspective. However, both internal and external commentators have noted that students’ solving of non-routine problems is rare – Andrews, 2009; Verschaffel et al. 1994, and Pauline’s lessons were no exception. Lastly, Emese’s practices seemed to accord well with how others, both insider and outsider, have viewed Hungarian mathematics teaching as one in which individual problem solving and collective sharing of solutions is commonplace – Andrews, 2003, Szalontai, 2000 – alongside emphases on mathematical reasoning – Andrews, 2009.

So, what can we learn from observing? We can learn much, but the danger of observing only within classrooms familiar to us is that we learn only to replicate what is current, the received orthodoxy. If we make systematic observations in cultures different from our own, we may acquire new insights that allow us to challenge that orthodoxy and, through careful evaluation, develop new ways of doing what we do. For us, seeing how our case study teachers structured their approaches to the teaching of linear equations leads us to a fundamental question; why do we, the English, continue to teach analytical approaches for solving arithmetic equations when our European colleagues all seem to start with the algebraic?

Paul Andrews teaches at the University of Cambridge. Judy Sayers teaches at the University of Northampton.

References


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