The main strand of the NRICH project is the NRICH website (http://nrich.maths.org), which was set up in 1996 to cater for secondary-aged pupils who were attending mathematics masterclasses at the Royal Institution. By 1998, problems aimed at primary children were included and the site began to provide for students aged five to 18-plus. At this time, the material was still principally used to meet the needs of mathematically ‘gifted and talented’ pupils. However, as the wealth of problems grew, teachers of lower-attaining secondary students began to use what were labelled as ‘primary resources’ in their classrooms. This, along with a growing body of research evidence, meant it became important to shift the website’s focus.

Mathematics enrichment and problem solving

My research began with looking into the notion of mathematics enrichment in the existing literature, both in the UK and abroad. Three themes arose:

- what is mathematics enrichment?
- who is mathematics enrichment for?
- where does mathematics enrichment take place?

There appears to be no definition of enrichment in the context of mathematics, but two features did emerge. Firstly, the majority of examples class enrichment as resources or activities extra to the standard curriculum. Secondly, enrichment is seen as a way to cater for some children — frequently the most ‘able’. However, I did find researchers challenging these assumptions. Piggott (2004) suggests there are two components to enrichment: mathematics content and an associated teaching approach (although, more recently, she has revised this to three, to include audience (Piggott, 2006)). At the heart of both of these is mathematical problem solving.

The purpose of learning mathematics is to be able to solve problems. In the classroom we want children to feel like mathematicians, which means teaching them ‘habits of mind’ (Cuoco et al, 1996) — developing their ability to think mathematically. Surely all children are entitled to such a classroom environment?

Low-attaining children in mathematics

In the context of my work, I drew on Watson et al’s definition of low attainment (2005), whereby this is used to indicate a current state, rather than implying anything about a child’s potential. There is a great deal of research which describes successful approaches to the teaching of low-attaining children where learners experienced not just being told facts and rules, but were encouraged to discuss ideas, use different methods and see links. These methods all have features of an enriching approach but it is evident that classrooms do not reflect these authors’ findings. This led to my research question:

How do problems taken from the NRICH website provoke mathematical thinking in low-attaining children?

Methodology

My study was undertaken from a social constructivist perspective and used a case study approach.

The ‘case’ was a pair of Y5 pupils, Lisa and Mark. I selected Y5 because I hoped they would be able to work relatively independently without the
**Figure 1: A numbered route**
Can you draw a continuous line through 16 numbers on this grid so that the total of the numbers you pass through is as high as possible?

```
3 5 2 7 8 1 3 2 0 9
1 3 4 6 0 9 1 3 4 5
7 8 2 0 8 9 1 4 5 7
2 6 8 1 6 1 0 3 0 5
3 2 8 4 5 9 7 6 5 6
6 1 4 1 5 4 3 6 7 9
7 5 7 2 6 8 7 3 8 0
6 6 4 2 0 0 9 2 8 7
9 7 5 4 5 9 4 0 9 3
2 3 8 1 9 0 1 4 2 8
```

You may start and finish where you like, and go horizontally, vertically or diagonally, but you may pass through a number only once and the line must not cross itself at any point. If you repeat this but only go horizontally and vertically and never go diagonally, what is the highest score then?

**Figure 2: Roll these dice**
You see here three dice; they're just ordinary dice and they are rolled in the usual way to see what numbers come up.

Two dice are similar and grey and the other is blue. Well, when they're rolled you get three numbers. What I'd like you to do is to add up the numbers on the two grey dice and then subtract the number that's on the blue dice.

So if one grey is 4 and the other grey is 5 and the blue is 3 we should add together 4 and 5 to make 9 and then subtract the 3, so that gives us a final answer of 6.

You could roll these dice over and over again and see what you get each time by doing the addition and subtraction. It might get rather annoying as the same numbers may come up over and over again and some numbers just might not appear. Like when you're desperate for a six in a board game, or any number if you need some special number to get from the starting square.

In this challenge I’d like you to consider:
- what are all the different possibilities that could come up?
- what are the final answers by doing the addition and subtraction each time?
- is there a good way of making sure you find all the possibilities?
- how will you record what you’ve found out?

Now have a go!

**Figure 3: Sets of four numbers**
Miss Brown was working with Becky’s group on numbers that share a certain property. She wrote 12 numbers on the board.

```
2, 3, 4, 5, 7, 9, 10
15, 21, 25, 28, 49
```

“You can all find a different set of just four numbers that go together,” she said, “And they must have a proper mathematical name. They can’t be just a set of numbers that you like!”

The children stared at the numbers. Alan put up his hand. “Like odd numbers?” he suggested. “That’s the right idea,” said Miss Brown, “but you can’t choose just odd numbers because there are more than four of them. You must use all the numbers in my list which fit your set. Anyone else got an idea?”

Becky put her hand up. “Numbers in the five times table? There are four of those.” “That’s right. But what would be a good name for them?” “Multiples of five?” suggested Becky. “Good,” said Miss Brown and she wrote on the board:

```
2, 3, 4, 5, 7, 9, 10
15, 21, 25, 28, 49
```

Becky’s set is multiples of 5

{5, 10, 15, 25}

There are 10 children in Becky’s group. Can you find a set of numbers for each of them?

Are there any other sets?

Copyright © 2003. University of Cambridge. All rights reserved. NRICH is part of the family of activities in the Millennium Mathematics Project, which also includes the Plus and Motivate sites. email: nrich@damtp.cam.ac.uk
pressure of national tests. I chose a pair so I could take into account their interactions. These two children were withdrawn from class on three occasions with another pair, Jane and Daniel. Lisa and Mark were chosen because they were in the lowest third of the year according to optional national tests and teacher assessment, and they would be willing to talk. On each occasion, the four children tackled a different problem, away from a computer, and taken from the NRICH website. These were A numbered route, Roll these dice and Sets of four numbers. Each problem was relatively ‘open response’, so the pupils could stamp their own mark upon them – I was keen to see what they could do, rather than what they could not.

My data collection focused on the mathematical thinking taking place, so I chose four different methods to get as much information as possible:
- participant observations as the children worked on each problem. I filled in an observation schedule based on activities typifying mathematical thinking (Jeffcoat et al, 2004)
- interview with the children’s teacher after showing her the video recording of one session
- focused interviews with the pair immediately after they had worked on each problem
- the children’s written responses to the problems (All the sessions with the children were audio- and video-taped, and the interviews were audio-taped.)

The analysis of my data was both qualitative and quantitative. Qualitatively, I focused on identifying what types of mathematical thinking had taken place and which part of the problem the children had been working on as this thinking happened. Analysis of the problems themselves was cyclic and ongoing as my research progressed. In this qualitative analysis, further themes emerged which led into quantitative analysis. This comprised looking at the frequency of occurrence of the different types of thinking which in itself gave rise to further qualitative analysis.

Findings, analysis and discussion

In trying to answer my research question, I had identified that mathematical thinking had indeed taken place – this was verified by interview with their class teacher, Mrs L, but to what extent was this attributable to the problems? I could have said that all the thinking I observed was provoked simply by the problems themselves, but observation data made me more interested in the immediate way thinking was provoked. This led me to identify four modes of provocation, see Table 1:

<table>
<thead>
<tr>
<th>Mode of provocation</th>
<th>Working definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>teacher</td>
<td>activity typifying mathematical thinking provoked by my input; for example, a question or suggestion</td>
</tr>
<tr>
<td>situation</td>
<td>activity typifying mathematical thinking provoked by a concrete occurrence which was out of the thinker’s control</td>
</tr>
<tr>
<td>self</td>
<td>activity typifying mathematical thinking provoked by something the thinker did have control over or by the thinker abstracting and/or assimilating the situation spontaneously</td>
</tr>
<tr>
<td>pupil</td>
<td>activity typifying mathematical thinking provoked as a result of something said or done by another pupil</td>
</tr>
</tbody>
</table>

Teacher-provoked thinking

This was thinking stimulated by me. For example, the following exchange took place as the children worked on A numbered route, and shows Mark sorting and comparing numbers in the route he has drawn as a result of my probing:

LW: Which ones might you want to leave out?
[ L looks over at M’s sheet. ]
M: [ Let’s look forward to look at sheet] Err … The smallest numbers?

Situation-provoked thinking

In these cases, thinking was instigated by a particular situation that happened to arise. For example, in the following excerpt, the numbers that Mark and Lisa happen to throw as they tackle Roll these dice cause Mark to compare the sum of the grey dice to the number on the blue dice and conclude that the results of the subtraction will be negative:

M: Oh no! We’ve got a minus.
[ M begins to draw a number line. ]
M: They ain’t got a minus [ referring to D and J as looks over at their paper. ]
L: Minus 2 [ points at M’s paper encouraging him to write it down ], minus 2.

Self-provoked thinking

This referred to instances when the pupil doing the thinking appeared to trigger the thought processes him/herself. While working on A numbered route, Lisa compares the numbers in her route with each other and the rest of the grid, indicating how she is striving to get the highest total. She volunteers this without any prompting:

L: [ Reading from her grid ] 9, 11 … Actually, yeah [ referring to the fact she will use a calculator to add them ].
LW: [ Chuckling ] Good idea!
L: Cos I’ve got really massive numbers. I’ve got a 1 and a 2, but that’s it.
LW: Well that’s quite good. You’ve only got one …

**Pupil-provoked thinking**

When analysing times where thinking was provoked by another pupil, it became necessary for me to step back and consider what had prompted this second pupil to say or do what he/she had said or done. This led to four sub-categories of pupil-provoked thinking as shown below, see figure 4:

For example, if the pupil provoking the mathematical thinking had been encouraged by me the teacher, initially, I coded it teacher-pupil-provoked or t-pupil-provoked thinking. The following example, taken from *Roll these dice*, shows my question prompting Lisa’s and Daniel’s responses, but Mark refutes their claim and tries to justify and explain his disagreement. In other words, he is provoked directly by the other pupils, but what they said was instigated by my enquiry:

LW: How could you get 13?
L: You could get 6 …
D: You could have a 6 …
M: No you can’t [standing up].
L and D: No you can’t.
D: You could …
M: You've already got the take away thing. And you can’t roll six with two dice and anyway the most [exact wording unclear here] you can roll is 12.

I conjectured that in the cases of self-provoked mathematical thinking this was triggered by the problem itself in conjunction with the pupil’s relationship with it – the child ‘digesting’ the problem. Mrs L agreed that this seemed to be true from the recording she watched. Self-provoked mathematical thinking might depend on the nature of the problem. This led me to wonder whether there was something about these particular problems that had encouraged the pupils to think for themselves. Similarly, situation-provoked thinking appeared to hint at something about a problem that caused them to think – a situation that came up by chance. Might some problems be more likely to instigate these situations than others?

**Quantitative data**

The aim of the quantitative analysis was to shed more light on certain types of thinking and make comparisons, rather than being statistically significant. Table 2 shows the ways each type of thinking was provoked across all three problems as a whole:

<table>
<thead>
<tr>
<th>Aspect of mathematical thinking</th>
<th>Teacher-provoked</th>
<th>Pupil-provoked</th>
<th>Situation-provoked</th>
<th>Self-provoked</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>exemiplifying</td>
<td>32</td>
<td>5</td>
<td>0</td>
<td>3</td>
<td>40</td>
</tr>
<tr>
<td>specialising</td>
<td>5</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>7</td>
</tr>
<tr>
<td>completing</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>correcting</td>
<td>4</td>
<td>13</td>
<td>1</td>
<td>14</td>
<td>32</td>
</tr>
<tr>
<td>clarifying</td>
<td>0</td>
<td>1</td>
<td>4</td>
<td>4</td>
<td>9</td>
</tr>
<tr>
<td>playing</td>
<td>9</td>
<td>0</td>
<td>0</td>
<td>9</td>
<td></td>
</tr>
<tr>
<td>comparing</td>
<td>38</td>
<td>15</td>
<td>4</td>
<td>11</td>
<td>68</td>
</tr>
<tr>
<td>sorting</td>
<td>91</td>
<td>1</td>
<td>0</td>
<td>5</td>
<td>97</td>
</tr>
<tr>
<td>organising</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>varying</td>
<td>7</td>
<td>1</td>
<td>0</td>
<td>8</td>
<td>16</td>
</tr>
<tr>
<td>altering</td>
<td>1</td>
<td>4</td>
<td>1</td>
<td>1</td>
<td>7</td>
</tr>
<tr>
<td>generalising</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>conjecturing</td>
<td>18</td>
<td>2</td>
<td>0</td>
<td>5</td>
<td>25</td>
</tr>
<tr>
<td>explaining</td>
<td>17</td>
<td>11</td>
<td>0</td>
<td>1</td>
<td>29</td>
</tr>
<tr>
<td>justifying</td>
<td>3</td>
<td>7</td>
<td>0</td>
<td>1</td>
<td>11</td>
</tr>
<tr>
<td>verifying</td>
<td>2</td>
<td>5</td>
<td>1</td>
<td>4</td>
<td>12</td>
</tr>
<tr>
<td>refuting</td>
<td>2</td>
<td>33</td>
<td>0</td>
<td>2</td>
<td>37</td>
</tr>
<tr>
<td>total</td>
<td>232</td>
<td>98</td>
<td>11</td>
<td>63</td>
<td>404</td>
</tr>
</tbody>
</table>

The most common mode was teacher-provoked thinking – over half of all the instances of thinking were provoked by me.

I then investigated each mode of provocation separately across the three problems to see which strands of mathematical thinking were most common in each mode.

**Pupil-provoked thinking**

In terms of pupil-provoked thinking, I found that the high instances of refuting and correcting were not attributable to particular problems. I would suggest that being in a non-intimidating environment and talking about what they were doing…
meant that the pupils were happy to engage with each other and challenge not just each other’s thinking but their own too. The high instances of pupil-provoked comparing came from *A numbered route*, which, as above, I would consider was due to the nature of the problem, and *Roll these dice*, where they didn’t agree with each other’s answers. This implies that the very fact they were working together provided a forum where making comparisons was necessary. If they had been working individually, the majority of instances of comparing in *Roll these dice* would not have happened.

**Self-provoked thinking**

In the case of self-provoked thinking, the highest proportion was correcting, and there were similar numbers of instances attributable to each problem. The recurrence of self-provoked correcting was due to the way the problems themselves are structured. Each one provides a context for knowing that a ‘right answer’ has been calculated, as later stages of the problem rely on this correctness; in *A numbered route*, it is accurate addition; in *Roll these dice*, it is correct computation of the numbers on the dice, and in *Sets of four numbers*, it is knowledge of factors and multiples. In this way, pupils were encouraged to be self-critical.

**Situation-provoked thinking**

The instances of situation-provoked comparing were all observed during *Roll these dice*. The pupils were repeatedly throwing the dice to get many examples of the possible results. Three of the four instances of situation-provoked comparing occurred when the total of the grey dice was less than the number on the blue, which prompted them to identify a negative total. The thinking on these occasions occurred only because they had happened to throw those particular numbers, and that was intrinsically linked to the way into the problem they had adopted. Taking this route had encouraged a certain type of thinking. To some extent in this case, the mathematical content of the problem also had a hand. Having realised that negative answers were going to be possible, conscious comparison of the dice became more integral to achieving a solution. In this way, another feature emerged as critical to the provocation of thinking – the mathematical concepts integral to the problem.

**Some conclusions and reflections**

I found that the three problems did cause my pair of low-attaining children to think mathematically and that these problems, taken from the NRICH website, might be used as part of an enrichment approach, having a place in the mathematical diet of low-attaining children. It could be argued that their mathematical thinking was provoked by the problems themselves. However, classifying the ways in which the mathematical thinking was provoked (teacher, pupil, situation or self) helped to reveal features of the problems that facilitated such thinking:

- the aim: specific aspects of mathematical thinking might be required by the very nature of a problem;
- the structure: allowing for multiple approaches and demanding application of early work to later stages in the problem in order to reach a full solution;
- the accessibility: level of content enabling an immediate route in, but later presenting barriers to be overcome.

My findings imply that by carefully choosing problems, teachers could develop certain aspects of mathematical thinking with all pupils. This, of course, has implications for my personal practice in terms of helping to devise problems for the NRICH website. If, as my research indicates, certain features of the problems might encourage particular types of mathematical thinking, it can be inferred that problems could be written which attempt to focus on specific strands. To some extent this process already takes place. In the last few years we have begun to choose monthly themes for the site which focus on aspects of mathematical thinking, as well as those which centre on mathematical content. In addition, we have been constructing ‘trails’ of resources aimed at developing a particular aspect of mathematical thinking (Piggott and Pumfrey, 2005, 2006 and 2007; Piggott and Brown, 2008). More generally, it would appear from my research that the particular structure of the three problems I used encouraged the pupils to become more reflective about what they were doing, as it was necessary to apply results from one part of the problem-solving process to a later stage. Therefore, developing more problems with this characteristic might help students to become self-critical in their approach to solving mathematical problems, no matter what their level of attainment at that time.

I would be very keen to hear from those of you who use NRICH resources with all your pupils – not just those who are high-attaining – please do get in touch.

**References**


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