On 29th June, 50 people attended the ATM East Midlands day conference on developing mathematical thinking. The day began with this address, which I wrote after reading some of the correspondence about mathematical thinking on ATM-mail.

A teacher of mathematics has a great opportunity. If he fills his allotted time with drilling his students in routine operations he kills their interest, hampers their intellectual development, and misuses his opportunity. But if he challenges the curiosity of his students by setting them problems proportionate to their knowledge, and helps them to solve their problems with stimulating questions, he may give them a taste for, and some means of, independent thinking.

G. Polya, How to solve it, 1944

This was written almost 60 years ago and yet it still remains a challenge for teachers of mathematics today. It is very easy to spend all your lessons teaching ‘routine questions’. The national curriculum consists of a long list of knowledge and skills that have to be learned and this seems to get even longer once it has been interpreted by the national numeracy strategy at KS1 and 2, the framework for teaching mathematics at KS3 and GCSE specifications at KS4. The SATs or GCSE exams by which your pupils – and you – will be assessed cannot, by their very nature as timed written tests, assess problem-solving skills or mathematical thinking in any depth. So why should teachers bother to teach in such a way that pupils can develop their ability to think mathematically if this is not needed for SATs or GCSE?

Fortunately, some teachers, think there is more to maths than doing well in tests. In fact, you may believe what some research claimed: that the more children are tested and graded the less motivated they become. In some recent research carried out by King’s College London and reported recently in the Guardian newspaper, pupils in Y6 expressed their bewilderment about the emphasis on tests by saying ‘We want to help each other, but we can’t now – it’s cheating’. No doubt you want to find time in your lessons to help your pupils enjoy mathematics, to understand its power and to want to know more about it, and maybe above all, to feel confident about their ability to do mathematics.

Further research on adults born in 1970 shows that their self-esteem and self-confidence at 10 was as important as their academic ability in predicting later achievement.

So your job as teachers is a crucial one: as Polya said, you have a choice between killing your pupils’ interest and giving them a taste for independent thinking. I hope by the end of today that you will have some ideas about answering these questions:

- What is mathematical thinking?
- What sort of tasks might encourage mathematical thinking?
- What can the teacher do to promote mathematical thinking?

Maybe the second question is the easiest one to answer. For example, today various tasks designed

WHAT IS MATHEMATICAL THINKING? Barbara Ball

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Maybe the second question is the easiest one to answer. For example, today various tasks designed
to develop mathematical thinking may be suggested to you. But your behaviour as a teacher is also crucial. What kind of questions will you ask? Which of your pupils’ questions will you refuse to answer? How much guidance do you give your pupils?

I remember David Fielker, a previous editor of MT, being asked how he taught his pupils to be good problem-solvers. He replied, ‘I give them a problem and I don’t tell them how to solve it.’

What about the first question? If you do not know what mathematical thinking is, how will you know whether or not your pupils are doing it?! To help us think about this question I am going to ask you to do a mathematical task and while you are doing it to pay attention to any thinking you are doing that may be mathematical thinking.

You should have four cubes. I am going to assume that these cubes have unit length edges, so that the surface area of one face is 1 square unit and the total surface area of one cube is 6 square units.

Now join two of your cubes together, face to face. What is the surface area of the shape you have made?

There are several different ways of seeing that this surface area is 10. You can count every square face you can see. You can notice that you have four identical faces of area 2 and two of area 1. Or you can argue that two cubes have a total surface area of 12 but that two faces, each of area 1, have been ‘lost’ when the cubes were joined together. I can well believe that there are other ways of seeing that the surface area is 10, which I have not thought of.

Now fit all four of your cubes together, joining them all face to face. How many different surface areas can you find?

After a few minutes there was general agreement from the audience that there were lots of ways of making a model with a surface area of 18, but only one way of making a model with a surface area of 16. (See figures 1 & 2.)

How do you know you cannot get a surface area of more than 18?

Because you must have at least three joins in order to join up four cubes. This means you lose six from the total surface area of four lots of six.

Is 16 the smallest surface area you can get?

Yes, because you cannot have more than four joins.

Is that obvious?

Now team up with the person sitting next to you, so that the two of you have eight cubes altogether. How many different surface areas can you find now?

Much discussion followed.

What is the biggest surface area you have found?

34 – when the model is like a ‘stick’. (See figure 3.)

Or when it is many other Nigel-shaped with the minimum number of joins. (See figure 4.)

What is the smallest surface area you can find?

24 – when it is a cube.

What other surface areas have you found?

All the even numbers between 24 and 34.

Why not any of the odd numbers?

Because you start with an even number for the total surface area (a multiple of 6) and you keep subtracting two for each join.

Who made a model of area 26?

The model was displayed by a member of the audience, but was discovered not to have an area of 26.

Is it possible to make a model of area 26 with your eight cubes?

Further discussion ensued.

I think I have convinced myself that you cannot make such a model. If you start with the two-by-two-by-two cube, with surface area 24, and remove any of the cubes it is easy to see that if you put it back anywhere else the surface area jumps to 28. But can I be sure that this argument is sufficient?

When I first worked on this problem with a class some years ago I had assumed that you must be able to get all the even numbers between the minimum and maximum values for surface area, and I thought one particular girl was just being a bit lazy when she submitted her report about models made from eight cubes, with illustrations of all the different surface areas except 26. I know better now!

These are the questions I asked my class to consider:

Consider any number of cubes joined face to face.

What is the maximum surface area possible?

What is the minimum surface area possible?

What surface areas between the maximum and minimum are possible?

What I like about this task is that the first question is easy – most students can offer an explanation of the result – while the second question is much harder, and I do not know the answer to the third question. And yet all three questions are easy to understand and do not require a high level of mathematical knowledge and skills, what they do demand is a high level of mathematical thinking.

This is my most recent attempt to summarise my ideas about mathematical thinking. The bubbles, all of which I haven’t yet managed to fill, describe what you do when you are engaged in a mathematical task: the thinking probably happens as you move along the links between the bubbles.
General questions and prompts

What follows is a summary from Anne Watson and John Mason’s book on mathematical thinking [1].

The words at the top of each box are the words the authors have chosen to describe mathematical thinking and underneath they have listed questions that the teacher might ask to encourage pupils to think in this way. What is really useful about this book is that the authors give lots of examples to illustrate what they mean by these questions.

Postscript

The feedback sheets from the delegates indicated that they felt the day had helped them to begin to answer each of the original questions. These are some of their comments on the various workshops they attended.

- Certainly made me think!
- Has changed my thinking about examples. The idea of using the particular to get to the general was inspiring.
- I always knew my art skills were poor - now I can add mental imagery to the list!
- Made me stop and think about what pupils get from the diet I give in terms of their thinking, rather than just their ability to pass exams.
- The day was brilliant. As well as all the useful and interesting ideas it was great for me at the end of my PGCE to meet a range of teachers still enthusiastic about teaching and maths.
- Challenging ideas, difficult problems. Good material to use with students.
- Excellent – inspiring – will I have the guts to go for teaching maths without a text book in my first teaching post?

Barbara Ball is ATM’s Professional Officer. You can find out about her latest activities on our website www.atm.org.uk or by contacting her at barbaraball@atm.org.uk

Workshops at the ATM East Midlands day conference on Developing mathematical thinking

Developing spatial imagery to support mathematical thinking George Knights
Structural approaches to the teaching of number patterns Paul Andrews and Heather Massay
The role of the example Chris Bills and Liz Bills
Maths without a text book Mike Ollerton
Exploring ‘teaching for mathematical understanding’ Viv Sloan
Mathematical thinking with a graphical calculator Wendy Führer
Geometrical thinking and dynamic geometry Barbara Ball
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