Introduction

"It is very easy for A-level teaching in mathematics to depend too much on exposition by the teacher and for students to adopt passive styles of learning. However, it is important for students in the sixth form as for pupils of all other ages to develop problem solving techniques, to pursue independent investigations and to discuss and communicate their ideas. It is by working in these ways that students develop the confidence which they will require in order to be able to make use of mathematics in their future studies and careers."

The Cockcroft Report

Particularly in recent years there have been attempts made within very many schools to move away from the traditional approaches to teaching mathematics, based on exposition by the teacher followed by exercises. Instead students are given greater responsibility for their own learning through problem-solving and investigational work, often tackled cooperatively in small groups. The establishment of GCSE has in some ways facilitated this change.

There is no reason why alternative approaches should be restricted to a particular age or ability range and, in particular, with students coming through to study A-level mathematics with a far greater experience of working independently it would seem wise to capitalise on this.

The availability of microcomputers and sophisticated calculators must also alter the way in which much of the A-level syllabus is taught and the relative importance of its various components.

Although the content and assessment of many A-level courses do not as yet positively encourage the development of fresh approaches, the aim of this book is to provide some ideas for teachers who feel that giving students autonomy is not only desirable but necessary.

Each unit of the book contains a problem, an experiment or a pattern to stimulate discussion and raise questions. Some are particularly appropriate for use at the beginning of an A-level course when encouraging students to develop relationships and work cooperatively (for example, "How True?").

The basic idea in each case, together with some suggestions for developments, are contained within a frame, and further comments about its use follow. It is expected that many of the ideas will be developed in very different ways by different groups of students, and the layout is designed to express this.

It is important to realise that many of the pages were not designed to be used directly as worksheets. It was felt that worksheets would provide too much guidance and could restrict development, and that it would be better for teachers to write worksheets appropriate to their own situations or to allow students complete freedom to follow up their own questions.
Many of the ideas can be investigated at different levels with or without knowledge of the content. They can be introduced at an early stage and returned to later.

The emphasis in the book is not so much on the ideas themselves, as on what questions they can provoke. Accordingly a few familiar problems have been taken and placed in a fresh context appropriate to students studying at this level (for example, "Where Will It All End?").

In many cases, particularly where practical work is involved, students should be encouraged to analyse the situation and predict results before trying something out.

Although some indication of the syllabus content which might be included is sometimes given, no attempt has been made to categorise the different units by content in case some possibilities are missed. For example, something which could easily be dismissed as just particle mechanics, such as "Dotty Problem", might also illustrate some aspect of pure mathematics.

The following people have contributed to this book:

Val Aspin
Alan Davies
Debbie Hawkins
Anne Haworth
Christine Hopkins
Yvonne Hughes
Peter Jones
Basil Reid
Dudley Ward
David Wynne
Limits

... different coloured hexagons

... odd and even numbers

As the pattern is continued what happens to the ratio between...

... black and white triangles

... volume and surface area

... squares and total length of lines
Notes on "Limits"

Approaches

Some of these patterns will doubtless be familiar to the students, but consideration of ratios will involve working at a higher level whether answers are sought by working numerically or by using various algebraic formulae. Certainly a whole variety of approaches have been used with these initial problems.

Students can be encouraged to pose their own questions about these patterns and in most cases there are ways in which the initial problem can be developed further. In the case of Pascal's Triangle, for example, it would be possible to consider multiples of three or numbers in modulo three.

Consideration of the relative importance of "missing" pieces within patterns can provide a useful link with the integral calculus. An example of this would be finding the volume of a pyramid by taking the configuration shown, then repeatedly doubling the number of tiers but halving the side lengths of the cubes, thus keeping the height constant.

Content

Summation of series, including arithmetic series and sum of square numbers; limits; ......

Reference

"Images of Infinity" Ray Hemmings and Dick Tahta (Leapfrogs)

In passing ..... 

Question: What is \( \frac{\sin x}{n} \) simplified?

Answer: 6. You just cancel the n's: \( \frac{\sin x}{n} = \sin \frac{x}{n} \)
Notes on "Strings in Equilibrium"

Approaches

It is helpful to have a brief discussion beforehand as to what might happen.

Practicalities

This can be set up by simply fixing one end of a string to the top of a door using a drawing pin.

Fix a piece of paper to the door and mark a line on it to keep the string at the correct angle. (Polar graph paper is particularly useful for this.)

A spring balance able to measure up to about 6 times the weight will be needed.

Extensions

Consider the relationship between the horizontal force required to keep the string at a given angle to the wall and the angle itself.

Content

Resolution of forces; Lami's theorem; functions; graphs; .....
Random Numbers

How could you use (a) coins (b) dice to obtain random numbers between 0 and 1 truncated to 2 d.p.?

The values of 200 random numbers are obtained. What sort of distribution would you expect if each value is:

- A single number?
- The smaller of two numbers?
- The larger of two numbers?
- A number taken as a power of 10?
- The average of two numbers?
- The average of ten numbers?
- The product of two numbers?
- The smallest of three numbers?
- The quotient of two numbers?
Notes on "Random Numbers"

Approaches

In considering the problem of generating random numbers students can be encouraged to take a simple case first - for example, integers between 0 and 4. It is sometimes more difficult for them to appreciate why certain methods do not produce random numbers than it is for them to accept valid methods.

The problem as stated is really one of obtaining random integers between 0 and 99. By tossing seven coins simultaneously it is possible to obtain random integers between 0 and 127 using binary numbers (head = 0 and tail = 1). Numbers over 99 can then be discarded. A similar method can be used with dice, but of course the numbers on a die do not go down to zero.

At some point it is helpful to discuss the difference between random numbers in the interval 0 to 1, and random integers between 0 and 99 divided by 100. Fundamentally, this is the difference between continuous and discrete distributions. Inevitably, if some of the suggested experiments are performed, the data will have to be grouped on some way and the distinction will become blurred.

In practice, obtaining random numbers using coins or dice would be tedious in the extreme. Tables of random numbers are available and some calculators have a random number button. If a group of say ten students are responsible for obtaining twenty values each, the distributions of quite large samples can be obtained reasonably quickly. Alternatively, a short program can be written for a microcomputer or programmable calculator, based on its random number generator. Even if students do not actually write the program, then certainly their involvement in its development will add credibility to the results. An example of such a program is given below.

```
10 DIM SUBTOTAL(20)
20 FOR P=0 TO 20
30 SUBTOTAL(P)=0
40 NEXT P
50 INPUT "SIZE OF SAMPLE",N
60 FOR R=1 TO N
70 X1=RND(1)
80 X2=RND(1)
90 MEAN = (X1+X2)/2
100 P = INT(20*MEAN)
110 IF P>20 GOTO 130
120 SUBTOTAL(P)=SUBTOTAL(P)+1
130 NEXT R
140 CLS
150 FOR P=1 TO 19
160 IF SUBTOTAL(P)=0 GOTO 210
170 HEIGHT = 100*SUBTOTAL(P)/N
180 FOR LEVEL = 0 TO HEIGHT
190 PRINT TAB(P+10,24-LEVEL);CHR$(255);
200 NEXT LEVEL
210 NEXT P
220 PRINT TAB(0,24)
```
The distribution of the maximum of two random numbers can be analysed in some detail starting with questions such as "What is the probability that both numbers will be less than 0.5?". Most of the other examples given can be discussed only in more general terms, but show how a variety of probability distributions can arise.

There are examples from other fields which make use of the 'best of three' or 'average score' and this enquiry gives some idea of the effects of using such results.

Content
Probability; continuous distributions; central limit theorem; .......

Mean Chain

6, 18, 12, 15, ...

Each term in this sequence is the mean of the preceding two terms.

How does the sequence develop?

What happens if one takes the difference between the two previous terms and divides by two?
Notes on "Mean Chain"

Approaches

It is possible to experiment with various pairs of numbers at the start of the sequence and see which pair provides the clearest view of what is happening. Then it has to be decided whether it is valid to generalise from this particular example. Alternatively, one can look at the general case and attempt to work out a formula.

There is a different approach involving the differences between consecutive terms which, again, can be used for a variety of particular cases, or with the general case.

Generating a sequence of this sort is not difficult using a calculator with a memory, and it is even easier if a short program is written.

Extensions

1. Sequences of vectors formed in a similar manner are the position vectors of points all of which lie in a single line. The sequence likewise tends to a limit.

2. What happens if the geometric, rather than the arithmetic, mean is used?

Content

Series; .....