How to use this book

The purpose of this book is to help you understand the thinking that lies behind this new approach to geometry and to let you consider the value it could have for your own students.

As the over-riding goal is the understanding of the mathematics, your active involvement and participation in the work is most desirable.

With this in mind, all figures that involve Tracings are repeated on the last page so you can copy them onto transparencies and use them with the appropriate Diagrams.

Having successfully done this, you will wish to see how these ideas work with students using an O.H.P. You will probably want to enlarge the Tracings and Diagrams on your photocopier as you copy them onto transparencies. Then everything is ready for you to introduce a group of students to your own choice of topic, or even the whole class.

First I suggest you attach a transparency by one edge to the O.H.P. This will enable you to show a transparency Diagram under it. Then a Tracing placed over it can be moved easily without disturbing the Diagram.

Suppose you choose Part 3.

Copy all the Part 3 Diagrams and Tracings onto transparencies. Have all these in two envelopes and you are ready to start.

Put Fig. 31 on the O.H.P. Get them to visualise what will happen if the circle is folded as shown. Do they all agree that if C falls on A, then D must fall on B? This could lead to much discussion.

Let it all come from them. Don’t suggest how they might look at it. Remember the sole objective is they themselves should think out what is happening.

Now move on to having Fig. 32 under the transparency flap. Place the Tracing of Fig. 33 on top of the flap, and over Fig. 32 (which gives Fig. 34). Ask them what will happen if the Tracing is given a 5 minute turn clockwise.

They should be able to tell you where points A¹, B¹, and C¹ will move to. Ask them how much more the Tracing must be turned before A¹B¹ is parallel to AD. Lead them on so that they discover that in fact \( \angle ADC = \angle ABC \) because both are equal to \( \angle A¹B¹C¹ \).

In Part 1 there is really no need for O.H.P.s, but in Part 2 they do help substantially in understanding the theory.

In Part 4 you may even find them necessary to make sense of the ideas being developed!
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Part 1
Covering a Chessboard with Dominoes

It is an elementary matter to cover the squares of a chessboard with dominoes, provided that each domino covers two squares exactly.

By using 32 of them we can easily cover all 64 squares in many different ways.

But suppose we take away two squares, the top right corner and the bottom left hand one, like this.

Now we ask the question: Can we cover the 62 that remain with 31 dominoes?

However much you try, you will find that you are never able to put in the final domino. Always you will meet failure . . . But is this proof that it is impossible?

To explore further the issues involved, let us look at the case of a 4 x 4 board.

Again we are unable to find a way of completing the modified board with dominoes. This time, the range of possibilities being much smaller, we come far more quickly to the conclusion that it is impossible.

But again this is no proof.

If we go further still and look at the modified 2 x 2 ‘board’, we see at once that it is definitely not possible . . . Only two squares are left, and they are only connected by a corner. But, more importantly, notice they are both black!

Now let us turn back to the 4 x 4 board. We see that it contains 8 white and 8 black squares. If the two opposite corners are removed, they must be the same colour (in this case white). This leaves us with 6 white squares and 8 black squares to cover with seven dominoes. But this is impossible, as it is clear that every domino must always cover one white square and one black square.

Once you appreciate this, you will realise immediately that every similar problem involving covering a square board (with its two opposite corners removed) with dominoes is impossible. Are you absolutely and totally convinced by this? Or, in other words, are you AATC by this?
The same type of argument can be used on other sizes and shapes of board.

We can start by considering square boards with an odd number of squares. For example, this 5 x 5 board obviously cannot be covered exactly by dominoes, because the number of squares is odd.

But, when you place in it 12 dominoes, you know what the colour of the square left over will be. Why? Because the 25 squares are made up of 13 black and 12 white, and (as the 12 dominoes must cover 12 black and 12 white squares) the one left over must be black.

Now consider these two arrays of squares. Although they are not coloured like those above, they could be.

Board b can be thought of as five squares containing 25 squares each. Suppose they are ‘chequered’ with an extra black square in four of them and an extra white square in the fifth.

Then thinking of the 125 squares altogether, there is a surplus of 3 black squares. As 125 - 3 = 122, this means that the maximum number of dominoes that can be fitted is only 61.

When dealing with any question like this, all we have to do is first of all think of colouring the squares like a chessboard, and count up how many squares you have of each type.
Covering a Chessboard with Straight Trominoes

As 4 x 4 = 16, and 16 ÷ 3 = 5 (rem. 1), it is clear that the most straight trominoes it is possible to get in a 4 x 4 board is 'five'. And you can soon convince yourself that the unfilled square will be in one of the corners.

Would you go about proving it like this?

“Well, it’s easy to fit in four trominoes this way, but then the fifth must leave a corner square when it is put in…”

But if we try filling a corner with the first tromino, and then always attempt to fill the next corner left vacant, we are led to the impossible situation where you cannot fit in the last tromino!

This proves it is impossible.”

Are you yourself absolutely and totally convinced by this?

But what about an ordinary 8 x 8 board?

We have already seen that the chessboard, ‘minus its opposite corners’, cannot be covered with 31 dominoes. The clue to a proof of this can come from noticing the colours of the squares, and observing that a domino always covers two squares of different colour.

Now in approaching similar problems with straight trominoes, can we use a similar method? We can do so only if we can make a chessboard of squares of three colours, so that each tromino definitely covers one square of each.

The question is then:

Can we colour an 8 x 8 board so that wherever we place a tromino on it, it will never cover two squares of the same colour?
As \(8 \times 8 = 64\), and \(64 \div 3 = 21\) (rem. 1), we see at once that the most straight trominoes we can get into a \(8 \times 8\) board is ‘twenty-one’. And one question we might then ask is: Which square is left uncovered?

Let us start by fitting in straight trominoes coloured ‘(a-b-c)’, starting with horizontal ones. It is obvious we must stagger them or we will have a’s underneath each other. Then we can continue by putting in vertical ones as shown.

Gradually we see a pattern emerge that provides a solution. If we letter the squares from the top left…

\[
\begin{array}{ccc}
  a & b & c \\
  a & b & c \\
  a & b & c \\
  a & b & c \\
\end{array}
\]

… (starting a new line whenever we need to), we see that we get a pattern of letters such that every row of three gives us three different letters! And similarly with every column of three!

Now we ask the question:

**How many squares containing each letter are there?**

If an a is in the top left corner, the ‘leading diagonal’ (shown in blue) contains 8 a’s, so we have altogether \(2 + 5 + 8 + 5 + 2 = 22\).

Meanwhile the number of times b appears is \(3 + 6 + 7 + 4 + 1 = 21\), and similarly the c appears \(1 + 4 + 7 + 6 + 3 = 21\) times.

As the letter that appears most is a, if we are able to fit seven straight trominoes onto an \(8 \times 8\) board, the additional uncovered square will be an a and thus must fall on one of the blue diagonal lines shown previously.

But equally well, we could have started at the top right hand square and worked to the left. So we can argue in exactly the same way that the uncovered square must fall on one of the set of dashed blue diagonal lines!

So, combining these two diagrams, we can conclude that the only possible positions of the uncovered square must be the four shown by the shaded squares in this final diagram.
What is a Proof?

“PROOF (Maths, logic.),” my dictionary states, is “a sequence of steps or statements that establishes the truth of a proposition.” For whom?

Clearly only for those capable of understanding the context and the argument developed. We started with the proposition: “that it is impossible to cover the remaining 62 squares of the chessboard with 31 dominoes.” The proof that I have given is only a PROOF to those who are able to understand and appreciate it.

Then consider the second proposition (not stated) that: “If 21 straight trominoes are fitted on to a chessboard, the square left uncovered will be one of the four shown in the diagram.” Once again do I provide a satisfactory proof of this? For whom? For you?!!

It is not so straightforward, is it? You may have been left with the feeling that “It seemed OK at a first reading, but I’d have to look it over again more carefully to be sure . . .”

But contrast this with what certainly used to be expected at the school level. To know a theorem meant to be able to reproduce its proof word-for-word. Understanding did not come in to it. After all, no marks can be awarded for understanding the theorem, as no attempt is ever made (in my experience) to test this.