The Editorial Board hope that:

- *Mathematics Teaching* is a space to share investigations into our teaching in some detail. Not generalising too much, but speaking for ourselves and of our experience. Never trying to speak for all.

- *Mathematics Teaching* honours the different ways that people know, do and engage with mathematics.

- readers will learn from engaging with the articles, in trying things out and in noticing the impact of their new practice.

- readers will engage in dialogue with the articles between issues.

- *Mathematics Teaching* gives teachers confidence to use ideas which are current and relevant to them in the ever-changing world of teaching and learning mathematics.

- articles do not shy away from describing the tensions, imperfections and challenges that are a part of teaching and learning mathematics.

The focus of articles may be:

- an account of something mathematical.

- an account of a visit to another classroom, another school, or another country.

- a description of using a favourite piece of equipment or resource.

- a reflection on experiences of different teaching and learning styles.

- views and news on current initiatives.

- responses to, or reflections on, articles from previous editions of *Mathematics Teaching*.

The Editorial Board are:

**Tony Cotton (Editor)** first joined the ATM in 1981. He has taught in secondary schools, spent time as an advisory teacher for anti-racist and multicultural education then worked for 3 different publishers before moving into teacher education. He worked in teacher education in Nottingham and Leeds before leaving the University sector to work full time as a writer and freelance educational consultant.

**Paul Killen** has worked extensively in secondary schools and Higher Education and is currently Head of Primary Programmes at Liverpool John Moores University. Over the past 20 years he has taught on a variety of initial teacher training programmes at primary and secondary level including PGCE, Undergraduate, School Direct and Masters programmes. For seven years Paul was Programme Leader for Teach First in Liverpool and delivered mathematics at the Teach First Summer Institutes in Canterbury, Warwick and Leeds. His particular area of interest is number and algebra and the links between these, especially with children (and students) who think algebra is something they cannot do.

**Jackie Salim-Smith** was a primary teacher for 25 years working across local authorities in the North West and has extensive experience of working from Reception to Year 6. For the past four years, Jackie has worked with School Improvement Liverpool, specialising in supporting schools in their mathematics provision. As part of this role, Jackie has presented on effective mathematics teaching to a wide variety of groups and audiences. She is a Key Stage One assessment moderator. Jackie’s specific area of interest is how manipulatives and concrete resources can be utilised with children of all ages to support and develop their mathematical understanding.

**Mahnaz Siddiqui** taught in Liverpool Primary schools for nine years becoming mathematics curriculum leader. Mahnaz moved to Liverpool Hope University (LHU) where she was involved in the design and delivery of both undergraduate and postgraduate initial teacher education (ITE) mathematics courses. Mahnaz worked at LHU for 14 years teaching mathematics education across a wide range of ITE courses and became the mathematics curriculum co-ordinator. For the past two years, Mahnaz has worked at Liverpool John Moores University. Her research interests relate to developing children's conceptual understanding, developing trainee teachers’ pedagogy, dialogic approaches and questioning, cultural diversity and inclusion. Mahnaz also has a keen interest in the power of Philosophy for Children (P4C), Forest School and outdoor learning.
## Contents MT266

<table>
<thead>
<tr>
<th>Topic</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Forewords by Tony Cotton</td>
<td>4</td>
</tr>
<tr>
<td>Danny's diary: The last two years by Danny Brown</td>
<td>5</td>
</tr>
<tr>
<td>Riding a bicycle: An analogy for understanding variation by Michelle Batstone</td>
<td>8</td>
</tr>
<tr>
<td>Observing lessons: A journey towards a professional development cycle by Alf Coles and Katherine Evans</td>
<td>10</td>
</tr>
<tr>
<td>Remastering mathematics: Mastery, remixes and mash-ups by Mark Boylan</td>
<td>14</td>
</tr>
<tr>
<td>A mastery approach: Taking the long view by Fiona Tidbury</td>
<td>19</td>
</tr>
<tr>
<td>A knot theory for eight-year-olds: Part 3 by Dan Ghica</td>
<td>21</td>
</tr>
<tr>
<td>Liverpool counts: Using mathematics to transform the lives of young learners by Tony Cotton and Jackie Salim-Smith</td>
<td>23</td>
</tr>
<tr>
<td>General Council report: The 2019 conference</td>
<td>25</td>
</tr>
<tr>
<td>Thinking mathematically everyday</td>
<td>26</td>
</tr>
<tr>
<td>Ethics and the mathematics teacher by Paul Ernest</td>
<td>28</td>
</tr>
<tr>
<td>Insight by Shane Johnstone</td>
<td>33</td>
</tr>
<tr>
<td>Mathematics borrows from everyday language by Jim Thorpe</td>
<td>34</td>
</tr>
<tr>
<td>Reflections on working with The Gang: A journey towards computational fluency? By Dave Benson</td>
<td>38</td>
</tr>
<tr>
<td>Drugs calculations: A non-formulaic approach by Lois Rollings</td>
<td>42</td>
</tr>
<tr>
<td>An appreciation of Trevor Fletcher: 1922 - April 14 2018</td>
<td>44</td>
</tr>
<tr>
<td>From the archive: The international commission at Krakow by Trevor Fletcher</td>
<td>46</td>
</tr>
<tr>
<td>Book reviews by Paul Killen</td>
<td>49</td>
</tr>
</tbody>
</table>

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The cover photograph was taken by Simon Carey. He can be contacted at sdcarey@gmail.com.
Forewords

by Tony Cotton.

Before I offer thoughts about the articles in this issue, I wanted to share the process of putting an issue together. There are five editorial board meetings a year. Between the meetings I will receive unsolicited articles. Any pieces that do not meet the criteria listed inside the front cover are returned with feedback and suggestions of alternative places they may seek publication. The editorial board discuss the remaining articles and decide whether and how we might use each piece. Authors are then informed about the decision and given feedback to develop the article. I am particularly keen to support colleagues new to writing and we are happy to spend some time with new authors developing a piece of writing.

We normally have 20-25 articles to choose from for each issue. This means we can make a careful selection so each issue contains a balance of pieces from across phases of education and from a range of perspectives. We also select articles that speak to each other both within issues and across issues. For example, in this issue the articles by Mark Boylan and Fiona Tidbury explore teaching for mastery from two different perspectives. Neither of these pieces imposes a view on the reader or tries to explain exactly what teaching for mastery might be. They take a more nuanced view exploring what the debate means for them, speaking from their own perspectives rather than imposing a view on others. “Never trying to speak for all”, as we write in our aims.

As I reread the pieces in this issue, I am reminded that one of the things I treasure about MT is the way in which pieces that may not initially speak of my experience often speak to my experience. Although it is a long time since I taught mathematics in secondary school, Danny’s diary reminds me of the importance of supporting learners in developing their own conjectures and examples. Similarly, I am reminded of how important it is to build in time and space for reflection, no matter how rushed I might feel. A quotation from this article, which I have pinned to my pinboard, is,

“Emotion … may be enhancing or inhibiting the flow of energy required for learning.”

This resonates with the two “mastery” pieces I mention above. Many primary teachers I talk to expend huge amounts of energy trying to find out if they are, “doing mastery correctly”. Danny Brown’s article alongside Mark Boylan’s and Fiona Tidbury’s offer us the opportunity to shift our energy into exploring what we might learn from the debate rather than trying to impose someone else’s solution on our teaching.

Paul Ernest reminds us that, on occasions, we find ourselves pulled away from our beliefs about the aims and purposes of our teaching. I wonder if it is at this point, when we are trying to implement pedagogical approaches that may contradict our deeply held beliefs about the purpose of teaching mathematics, that we become frustrated with teaching. For Paul, teachers have twin ethical responsibilities. Firstly, to those that we teach and to all whom we share the planet with. This responsibility means that we should be honest, respectful, caring and supportive to our learners and to all we come in contact with in our roles as teachers. Secondly, we have a responsibility to our profession. This responsibility includes supporting colleagues, working collegially within our institution and supporting and enhancing the profession. I found this article useful in helping me think of the ethical role that we have as the editorial board of MT and offered me another way to evaluate each issue.

I found the use of the Hippocratic oath, “do no harm” in Paul Ernest’s piece a useful link to Lois Rollings’ discussion of the ways in which nurses are taught and learn to calculate the quantities of drugs they should administer. Accuracy in calculation and the ability to know if an answer is reasonable are vital.

A responsibility I take seriously is to honour the memory and tradition of both the ATM and MT. Trevor Fletcher, the first editor of MT, died last year. This issue contains an appreciation and reproduces an article he wrote over 55 years ago. In it he explores what we can learn from close observation of teaching. Or, as we seem to be calling it currently, lesson study.

Postscript: Whilst we do receive lots of articles, we would love to hear more from colleagues working with very young learners. Do get in touch even if your idea is only in its gestation phase. We will work with you on developing these ideas for the readers of MT. We also welcome feedback and print any questions or comments that are sent to us to develop the conversations between issues. Contact the team at journaleditor.atm.org.uk.
Danny’s diary: The last two years

In this diary, Danny reflects on the last two years of his teaching practice and identifies some pedagogical principles.

In 2017-18, I taught a group of 8-11-year old children once a week. Before teaching them, I observed them working through a scheme, in four differentiated groups. The lesson was about written addition and subtraction methods. The teacher showed a method to each group in turn and set them exercises to work on. After completing their work, the children brought it to the teacher to be marked, who used the teacher guide and a red pen to tick or, if incorrect, to circle, their work. At one point the teacher gave me the guide and the red pen and invited me to do the marking. When the first child came to me, I gave her the pen, without the guide, and invited her to mark her own work. I noticed that one of the answers was incorrect and started to wonder what might happen. She ticked them all correct apart from that one, hesitated, changed the answer to be correct, and ticked it.

I observed the children using algorithms to answer questions. The mistakes they were making indicated to me that we would need to work on ways of understanding the mathematical structures underlying the algorithms. As for evaluating their work, I believe that an important part of learning mathematics is to develop the ability to make, test and modify conjectures. I wondered what might happen if these learners were given the opportunity to generate their own problems, work on them collaboratively, and evaluate whether they thought their answers were correct. Further to this, I hoped I might be able to convey to them something of my enjoyment and enthusiasm for mathematics.

Over the following months, I explored various ways of working with these children. I found they were enthusiastic when invited to generate problems of their own for each other to solve. I began to create situations in which the children’s energy might be released, which often involved me stepping to the side of the process for a while, from which position I might step in, in order to channel the children’s energy towards relevant actions. This worked best when I had thought carefully about constraints to the tasks and activity.

For example, in one lesson I invited the children to create a calculation with the answer 1.2, and then make one number in the calculation blank, thus forming an equation to be solved. I had found that all the children enjoyed writing their questions on the whiteboard. This was no exception. The children gathered around the board to write their questions (figure 1).

Figure 1: Children write their own questions.

They then worked on each other’s questions and started to fill in the blanks on the board, sometimes modifying each other’s conjectures (figure 2). When the children reached a consensus on the answer, the equation was erased from the board. There were a couple of equations left at break. However, a number of children gathered at the board and continued to try to solve them. There was one equation left at the end. The child who created it wrote her name above it and the words, “Still remains unsolved”.

I started the next lesson by inviting the children to continue working on the unsolved question. After it had been solved, I drew their attention to an inconsistency from the previous lesson: \(0.3 \times 0.4 = 1.2\) had been accepted by the children as correct and erased from the board, whilst \(0.2 \times 6 = 1.2\) had also been accepted. They modified the first answer, and then they spent...
some time generating and working on questions, “like this”.

During 2017-18 I also taught two 17-year-old students, J and K. When I started teaching them, I experienced them as reluctant to initiate actions. I tried various approaches to foster independence, which mirrored the approaches I was using with the primary students.

I taught J and K to use software such as Geogebra and Excel to generate many examples and use these examples to make and test their own conjectures. On one occasion, I invited them to explore the effect of changing $a$ and $b$ in the recurrence relation $u_{n+1} = a.u_n + b$ by creating a spreadsheet. After doing this for a while, K said, “$a$ determines whether there is a limit or not,” and then made the conjecture that $a$ must be less than 1 or greater than -1 for a limit to exist, which they tested using their spreadsheet. At one point they became visibly excited and called me over to look at the spreadsheet. They showed me the boundary example $u_{n+1} = 1.u_n + 1$, with $u(1) = 0$, which alternates between 0 and 1.

I found that an essential part of this process was to provide lots of opportunities for reflection, more than I had previously in my teaching. J and K found it useful to make notes of examples and experiences that were meaningful for them, for example, a question they had found difficult or enlightening, or a summary of what they had found out. I encouraged them to do this spontaneously, but they rarely did so, although the quality of their notes improved considerably. This was also the case with the primary group. They kept an exercise book for reflection only, a physical example space, which proved to be a valuable source of reference for them.

Moving from directed to spontaneous action with J and K was a lengthy process. It seemed to me that J and K had become reliant on the teacher telling them what to do. In an attempt to move towards spontaneous action, I introduced prompts that were then faded out over time (see Mason, 2007a). Prompts that they adopted included, “Of what is this question an example?” and, “What do I know? What do I need?”.

It is hard to know to what extent these prompts were useful, but I did experience J and K becoming able to initiate mathematical actions more frequently.

On one occasion, J came in with a question he had created at home for me to solve, without being prompted. I felt surprised and excited, and exclaimed, “My work here is done!” His spontaneous creation of a problem for me to solve was indicative of my feeling that he invested more of himself into our work together than K, and their results reflected this. It was evidence for me of how the teacher can only offer opportunities, and that other aspects of the learner’s self, in relation with the teacher, may block the flow of energy.

2018-19

I have continued to refine these approaches in my teaching this year. I recently worked with a small group of first-year mathematics and education undergraduates. During a session on rational functions, I started with the following set of questions, offered sequentially in time:

Give an example of a function with a horizontal and vertical asymptote...

... and crosses the x-axis at $x = 2$...

... and has a vertical asymptote at $x = -7$...

... and a horizontal asymptote at $y = 3$.

This technique provides a structure for generating examples at each stage that can be compared for sameness and difference. This set of questions had the additional surprise of having only one final solution, thus allowing the students to evaluate their solutions via consensus. I later invited them to create a function with two vertical asymptotes and wrote their responses on the board (figure 3).

I became excited on recognising the richness of the examples but kept it to myself, so as not to deprive them of the realisations I hoped they would reach. I invited them to test their conjectures using Desmos. I heard a few “Aha!” sounds, perhaps due to the beautiful curves, perhaps due to the realisation that some of the functions did not have two asymptotes. There followed a discussion about why this was the case. The student who generated the boundary example $\frac{x}{x^2-2x+1}$ made the connection between the denominator having one root and the function having one asymptote, upon which the students spontaneously started calculating the equations of the other asymptotes and checking them against their graphs.

A few weeks ago, I was invited to hold a one-off session with a 14-year-old via Skype. I was not sure what we might achieve in one session. I started by asking him about his experience of mathematics and he said that he had lost confidence. He wanted to work on linear simultaneous equations. He showed me some of his class-work on them, which was correct, but said that he could not remember anything now about how to do them.
Although I had a short time to work with him, I felt that I might be able to engage him so that his confidence in his own actions could grow and draw his attention to some tools for reflection that might be useful for improving his ability to remember. The following is an example of this.

I invited him to create a pair of equations of his own. I was unsure whether this was a good idea as any equations he would create might have tricky solutions, possibly eroding any confidence he had gained. I felt the risk was worth it to provide him with an experience of confidence in his own actions.

When I asked him to create a question, he said, “I’ve not made questions up before, I’m not very good at it.” He then went for it, offering the following pair: \(10x - 3y = 20\) and \(5x - 2y = 10\). He solved them correctly, without any prompting. I experienced a tingling feeling on the back of my neck, a release of endorphins. I asked him how he felt, he said he felt good and started smiling.

**Conclusion**

Over the last two years, I have identified some principles for my practice:

- Share the creation of examples between teacher and learners. Inviting learners to generate examples often results in a release of energy.
- Explore scaffolding and fading through mathematical prompts in order to help learners move from directed to spontaneous action. Learners will adopt prompts that are meaningful for them.
- Create a culture in which answers are conjectures that can be tested and modified by the learner as well as the teacher. For example, allowing learners time to evaluate their own work, or through using computer software.
- Respond to what the learners do, what they find interesting and challenging, both in the moment, and in the design of lessons and sequences of lessons.
- Allow time for reflection, through discussion, or encouraging learners to make notes of examples that are meaningful for them. I see this as the creation of an example space, within which boundary examples are particularly significant.
- Consider how various aspects of my and the learners’ selves, such as emotion or will, may be enhancing or inhibiting the flow of energy required for learning. Working with inhibitions to the flow of energy may take time or could be beyond my ability: I can only provide opportunities.

**Reflection**

The approach outlined here is one of a range of pedagogical approaches laying on a spectrum: To what extent does the teacher determine what learners will work on? Whilst there is some freedom in the choice of examples, I found that it was important to carefully constrain tasks and activity in order to channel activity towards relevant action. At the heart of these pedagogical choices lies the didactic tension:

The more precisely the teacher specifies the behaviour they are seeking or expecting, the easier it is for learners to display that behaviour without actually generating it for themselves. (Mason, 2007b)

I regularly sought feedback from learners about their experience of working in this way. The vast majority of learners were positive, but a few students were less so, one saying she was, “Confused about the purpose of what we are doing.” Asking learners to generate examples can bring frustration for learners. Some of this frustration may be a natural part of learning mathematics and may in part be due to the learner feeling that the teacher is not fulfilling their part of the didactic contract. The teacher sets the tasks, the learner learns by doing the tasks. For me, this frustration can bring doubts to the surface. I often wonder whether the learner’s frustrations are more or less productive, whether to offer more or less support.

How do I know what support to offer, and when? I was struck by something that happened this week in my daughter’s swimming lesson. Most of the time she uses a noodle-float, but sometimes the teacher takes the float away. Then the teacher supports her with her hand as she swims, describing the technique and offering encouragement. This week the teacher took her hand away, just an inch or two. My daughter was a metre or so from the edge of the pool. Her arms and legs started paddling frantically, managing just to keep her head above water. She reached the side and grasped it. The teacher turned to me with her eyebrows raised and smiled. I wondered how she decided it was time to take her hand away?

**Danny Brown is at home with his daughter, training as a counsellor and continues to be involved in mathematics education.**

**References**


I am grateful for the help of Tom Carson, John Mason and Mike Ollerton in the writing of this article.
Riding a bicycle: An analogy for understanding variation

Michelle Batstone reflects on variation, inspired by a session at the South West PD providers’ conference led by John Mason and Anne Watson.

I have not ridden a bicycle for about 10 years but, to use current terminology, I would argue that I have mastered riding a bicycle and would confidently describe myself as a bike-rider. Being born in the early 1960s, I am of an age where it was thought beneficial to attach stabilisers whilst you learnt to ride. This was coupled with verbal instructions from my father such as, “When the pedal comes to the top, transfer your weight to that pedal and push down”; “don’t forget to look where you are going and make sure you know where the brakes are.” However, as we now know, learning to ride using extra wheels removes the first key skill that a child needs to master, the art of balance. In fact, stabilisers teach children to use their bodies and their weight in ways totally at odds with what is needed to ride on two wheels. Often, when cornering on a bike with stabilisers, with one training wheel on the floor, the rider will naturally shift their weight the opposite way. Not what you want to do when cornering on two wheels. Thankfully, lightweight balance bikes were invented as a “first bike”, teaching a child how to use his or her body to control a bicycle.

Setting off; moving on two wheels; turning and stopping are the fundamental skills a cyclist needs. Balance bikes are cleverly designed to support children to discover how to do all these things by themselves and without instruction. Most balance bike graduates will find moving onto their first pedal bike a relatively quick and painless experience. By now, you might be wondering what this analogy has to do with variation in mathematics.

Let me begin with my original understanding of variation introduced to the teaching community recently through the NCETM as one of the “5 big ideas of teaching for mastery” (see https://www.ncetm.org.uk/resources/50042). I have attended NCETM regional residential training that has proposed a particular view of variation as being either “procedural” or “conceptual”. To summarise, they posit that “procedural variation” occurs when one plans for small steps in lesson design with slight variation to an activity, a practice exercise or within the process of solving a problem. “Conceptual variation” they suggest consists of understanding mathematical concepts from multiple perspectives. This illustrates the essential features of a mathematical idea, using different forms of visual representations to extract the essence of the concept and to recognise it in any context. However, as I listened to John Mason and Anne Watson, I experienced what can only be described as an “a-ha” moment. This eureka episode of suddenly understanding a previously incomprehensible problem or concept became the impetus for my bike riding analogy.

Although variation is not a new idea in mathematics, these latter terms have been introduced by mathematics educators in China trying to articulate the ideas underlying their approach to teaching mathematics in ways that are accessible for Western audiences (see Gu, Huang and Marton, 2004). However, it is these types of superficial distinctions, such as those made between “procedural” and “conceptual” variation, that represent an additional bolt-on (the stabilisers) to keep the pedagogy of mastery (the bike) upright, that I began to question.

This personal learning journey of questioning was provoked by John Mason and Anne Watson through thoughtful mathematical puzzles that required the audience to describe patterns they had noticed; to predict and to generalise whilst working on the mathematics. I began to appreciate that variation in
mathematics was, quite simply, to draw attention to the underlying relationships that the teacher wishes learners to notice, but also one that depends on learners having a disposition or habit of mind, to wonder why.

Their final prompt to “name that variation” was not about the number of examples, but their choice of numerical examples or images that prompted us to consider the role of variation beyond a surface distinction between procedural and conceptual. These drew our natural powers of motivation to go “with the grain” as we appeared to be completing the activity by following an obvious sequence, whilst the real aim of the task was designed to independently explore and gain an understanding of what was happening with the relationships by going “across the grain”, all without their direct instruction.

Our pedagogy on the effects of variation on mathematical experiences was developed by their prompt for reflection.

The balance bike

A good quality balance bike is the object that teaches a child how to use his or her natural powers of body balance to control a bicycle because children are invited to focus on balancing, rather than pedalling. In my opinion, the ideas and notions of John Mason and Anne Watson are the balance bikes of variation in mathematics. Rather than trying to pick apart variation as a pedagogical practice that teachers need to bolt on to their planning like a set of stabilisers, one of the key messages I took from their talk is that we should be selecting tasks that provoke learners into using their “natural powers” to detect patterns, to imagine and express predictions, to generalise. Therefore, an aim of mathematics teaching could be to offer careful mathematical examples that connect with learners’ natural instincts to follow patterns of numbers or sequences related to numbers by following an obvious sequence “going with the grain” and also to invite learners to connect with their natural powers to understand what is happening “across the grain”.

The effects of variation relate to how learners engage with the task, do learners notice what we hope they will notice? Therefore, a task that invites the learner to explore and discover an underlying mathematical structure is, in my opinion, the essence of variation. This practice will foster mathematical moments of personal insight in learners that I believe are at the heart of deep conceptual understanding.

This reminds me that a task which does not permit and encourage learners to use their own powers to encounter core mathematical themes, is not a pedagogically effective task. If we ignore this message, we are missing the opportunity to support our learners to be masters in mathematics.

Michelle Batstone is a lecturer in mathematics education at the Institute of Education, Plymouth University and teaches at Halwill Primary School.

The presentation from the conference is available at http://www.pmtheta.com/jhm-presentations.html#.

References

Lesson observation has a history within ATM. In MT265, David Fielker described a practice that was used in the early days of the ATM of demonstration lessons (see also From the archive in this issue). We offer here a story from one school which developed a way of using observations for their own professional development, around a focus on students mastering mathematical concepts.

The context of the work linked to the National Centre for Excellence in Teaching Mathematics (NCETM) professional development materials for teachers. The addition and subtraction elements of the primary resources are now complete and freely available at https://www.ncetm.org.uk/resources/50639. The school we focus on in this article was trialling the materials, having not been involved in their creation.

The materials aim to offer a detailed and conceptually coherent route through the mathematics curriculum. The documents offer a “spine” to guide teachers' work in mathematics and are perhaps unusual in:

- the detail of the conceptual breakdown of content.
- offering a new ordering of the curriculum.
- promoting new pedagogical strategies, linked to an emphasis on offering all learners access to the same content.
- offering a consistent use of key representations for mathematics throughout the primary years.

The materials aim to offer ways in to content that will be accessible to learners and that help build conceptual understanding. The materials are aimed at teachers and, vitally, work is needed to translate them into lessons. Our experience is that this work needs to be done collaboratively.

Towards a model of professional development

The work we report on here took place in a primary school in Bath. The school implemented work on the spine materials using a professional development model that they developed of: co-planning, co-observation and reflection.

Co-planning would begin at a staff meeting, focused on particular spine “segments”. Teachers would prepare a week’s work and part-way through the week would co-observe each other. There were de-briefing sessions immediately afterwards. Some of these teaching sessions were also video-recorded. In a subsequent staff meeting, small clips from a lesson would be worked on by the staff, identifying teaching strategies. The model can be pictured as in Figure 1.

The cycle proved feasible for the school to implement in terms of cost and appeared to open up conversations for staff across the school, for example, to discuss and agree on common representations to use in their teaching of mathematics.

There were two elements of observing lessons, one live, followed by an immediate reflection meeting and one based on a video. We focus on these elements and offer some detail of what took place.

Figure 1: A cycle of professional development.

Co-observing lessons

The foundation stage observed a year 1 lesson and vice versa, in November 2017. The year 1 lesson was based on segment 1.2 (see, https://www.ncetm.org.uk/resources/50719), “Introducing ‘whole’ and ‘parts’”. The de-briefing discussion that followed involved the teachers from the foundation stage, including Katherine, the year 1 teacher, Alf and the mathematics lead for the school. The following exchange is from field notes and is chosen to exemplify the focus of the initial discussion.

Katherine (K): We looked at the spine together in the staff meeting and M [the year 1 teacher] chose what she wanted to focus on that would link in with her National Curriculum targets and objectives. Then we worked out what the concepts were that we would need to explore with the [younger] children to link into that, what the foundation was
that this was building on. Which was this idea of “part” and “whole” and actually understanding the concepts and the words themselves.

M: When you say to them, “What does the word whole mean to you?”, S immediately said, “It’s a dark place where animals live.” So that’s where they were coming from. They didn’t really have the concept of any other sort of whole/hole at all. And as the week’s gone on that has still been a little bit confusing for them, in that something could have a hole in it, and because it has a hole it can still be whole. But the misconception was that it’s not whole because it has a hole … I showed … a circle and I was expecting then to say that’s whole, but no, because it had the tiny little hole in the middle they were then thinking that’s not complete.

K: I think that’s a fundamental part of being able to understand what a whole group is, being able to take a number and think that’s my whole. They need to be able to understand that whole concept, that different numbers can be a whole. That 12 can be a whole that you can split into 10 and 2, but then actually 10 could be a whole group. The more we talked about it the more we realised how complex it is.

J: Yeah, I mean a tiny part is missing, but it’s whole because there’s not a part that’s missing, it’s because that’s how it was formed originally as it’s whole. That’s so conceptually difficult.

M: We were talking about shapes as well. I can have a square on a piece of paper and I can cut a line up the middle of it and say now I’ve got 2 rectangles, but if I cut a circle in half straight up the middle then that’s a semi-circle, or half a circle, that still relates to the circle.

L: I think with our few days that we’ve been teaching we’ve found these really interesting misconceptions that have come up. So the size of a whole, one of the children was really set on the idea that a whole object had to be old.

This transcript is one instance of something that has recurred at different stages of the cycle, which is all of us coming to realise that mathematical concepts are more difficult and complex than they might seem. In the extract above there is a focus on a conceptual questions, “What is a whole?” and “What is a part?”; that perhaps might have been glossed over in the past.

There is nothing in this conversation about evaluating the lessons or the teaching itself. We find this striking and surprising as there are many research studies that suggest that a focus on the mathematics is rarely present in de-briefing from lesson observations.

Perhaps of relevance here is that the school had, in the previous academic year, implemented a programme of “Dragon Lessons”, where up to six or seven staff would observe one lesson and de-brief afterwards, with the person teaching rotating throughout the year. In other words, there was some experience already present within the group, in relation to working with each other following a joint observation. Katherine’s prompt, above, started the discussion and we see this as significant, in orienting discussion towards the concepts of “part” and “whole”. We notice a sense of the teachers empathising with the complexity of the language, that something with a “hole” can still be a “whole”, yet other times, taking a “hole” out of a “whole” means you are left with a “part”, and at other times still we might switch to think of the part you have taken away as a new “whole”. The immediate outcomes of such thinking are not in the realm of “tips for teachers” but more about a shifting orientation towards becoming curious about students’ thinking and searching for the logic in viewpoints that we do not share.

Working on video

After the first cycle of lesson observations and reflection meetings within the three teams, there was a meeting with all staff, where we worked on a video recording that had been taken of the year 5 lesson. Two lessons had been video recorded and Alf suggested a clip of three to four minutes that could be worked on, length based on principles of working with video (see Jaworski, 1990 and Coles, 2013). Alf chose a section of whole-class interaction. Alf began by getting participants to reconstruct what took place, without moving to evaluation or judgment. A key distinction that is required of people in the meeting is between an “observation” and a “judgment” (or “interpretation”). An example of a judgment could be, “At the start of the lesson the boys were disengaged”, a judgment because it is not clear what criteria are being used for “disengagement”, or indeed how we could ever tell this was the case. Turning this judgment into an observation might lead to a statement more like, “Four girls answered all the questions”.

We offer, below, a shortened version of the discussion that took place. The transcripts are taken from field notes and have been edited to try to give a sense of the flow of the conversation. We intersperse the transcript with some commentary. The teacher on the clip is J.

Alf (A): Where did it begin? What happened first of all?

S: Two children were given tasks to do.

C: They were talking about different representations. I’m trying to remember if J used
the word representations.

A: So J begins with something about representations. What happens next?

K: If we’re right about him asking how many representations, I think it was that, and then a child came up and did something, a bar model I think, and then said about the cherry, which he drew on a smaller board because there was no space.

H: But J listed some different kinds before that, I think J said column, addition, bar maybe.

A: So there was a listing of some methods?

K: I remember J saying “I don’t think you’ve got space for a cherry model on the board so here’s the smaller board with extra space”.

A: So at some point J says there’s not enough space on the board and there’s a smaller board produced.

S: That’s weird.

A: Did you not see that?

S: I saw two children at the front.

C: Did J say, “Does anyone have any other representations?” And D said, “Cherry”, and then J invited him to come and show the cherry but there wasn’t space on the board so he showed it on the little one.

H: That sounds about right. And then J also said, “In a minute, for these children I’m going to ask you to repeat this”, or, “See what you think of it”.

Up to this point, the group have been disciplined in focusing on the detail of events, sharing their sense of verbatim quotes from the video and also when they have disagreements about what took place. Alf’s role was largely repeating. Discussion continued in a similar way. Alf is alert to bringing out potential differences in what was seen or heard, as a motivation to re-watch a section of the clip, perhaps after five to ten minutes with a focus on some specific questions. Following a re-watching of a section of the video, discussion continued much as above, with a focus on the detail of what was said and done. Alf then moved discussion on to a later part of the clip.

A: Anything more at the end? The kids were possibly writing. Sue was saying there was a bit where she thought they were very quiet at one point, looking.

R: Are they also interested in what the children on the table were doing. Was there some sort of observation going on with the children at the table?

A: I guess we can’t tell what they’re interested in, but we can say what they did or said.

Here, Alf reminded the group of the discipline of staying with the detail of observations, trying to avoid the interpretation. To support the focus on observation, the group re-watched the start of the clip again.

A: So you were seeing some interaction on the carpet. Let’s go back and watch the beginning. It seems like we’ve got quite a few questions about the beginning and particularly what was J saying and what were the students saying? See if that helps.

M: I missed the first time that you’d said column, line, bar.

A: So where does that come? Where does J say that?

M: About twenty seconds in maybe.

K: Does J say, “What different representation”, or, “What other representations do you have on your boards” and then L put his hand up and said, “Number line”.

The sense, above, of teacher M commenting that they had missed something on first watching the video, when they came to re-watch it, is a familiar reflection from others using this discipline of trying to re-create a short clip of video. There is an immediate point here to reflect on as teachers, how on earth do we notice what is happening in real time in a classroom, when we can struggle on a video to know what was said on multiple re-watchings.

The second part of the way of working on video, having spent maybe 20 minutes on the re-construction phase, is to move to interpretation around an agreed focus. For this group, Alf prompted a focus on teaching strategies to support (their interpretation of) “mastery”. We noticed a significant difference in how the teachers were now talking about the lesson.

H: Trying to use different representations for the same thing to support children who might learn in different ways, using different methods.

A: So giving children choice.

K: Using technical vocabulary, specific vocabulary.

M: Peer modelling.

H: A variety of representations. Some maybe appealing to visual representations rather than written.

A: How would you characterise those different representations?

H: Bar, written methods, column, line maybe more of a scaffold method.

Alf’s role here seems to be one of sometimes naming
succinctly a strategy that has been described and sometime pushing for more detail.

A: Other things that you saw happening in the classroom.

H: Were the two methods working with the same numbers? So they were in a way checking their own work, hoping that both methods would lead to the same answer.

A: So the focus isn’t on the answer, in a sense they’ve got the answer. The focus is very much on methods. Anything else?

K: J was genuinely interested in what L was doing as well. I found it quite authentic, the response. It wasn’t fake and I think that was why the children were so focused, they could see it was something special. It felt like something special in the room.

J: There is a point, I just realised watching the video, with some children near me by my feet. They were talking and I told them to be quiet, it’s important and then I stare at L for a bit ... I knew it was going to take him a while and be hard, so when he manages to muddle through it completely individually, there’s something quite special happening. And when I realise he’s going to get there completely on his own, that’s when I turn my focus away from everybody else and put it all on him.

A: In that sense you’re modelling a focus on him to the children. It’s interesting when we focus on the questions that we ask as teachers, which are often “guess what’s in my head”, in a sense they don’t feel like particularly interesting questions. But a focus on methods, not the answer, feels like you genuinely don’t know what they’re going to say and you’re genuinely interested.

It is apparent that a range of strategies came out of discussion. These were not planned in advance and the agenda for the session was on teachers finding something of interest to them and articulating some practices they could try out, without an attachment to what those strategies were. There is an interesting parallel however, in the focus on concepts and children’s thinking from the de-brief conversation and the sense in this final piece of transcript of a focus on teaching strategies that support a focus on students’ thinking in the classroom.

Reflections

We hope some of the power of the lesson observations that took place comes through in the transcripts above. As a profession, it seems we do not know much about the conditions under which reflective conversations lead to new insights. As we (the authors) reflect on the work that took place, we see evidence for the power of a focus on mathematical concepts. This focus is supported by the NCETM materials, which themselves are largely organised around conceptual development. We also see a power for making a distinction between observation and interpretation, when working with video, and being explicit when each one is required. We suspect there was something significant also in this process having been designed within the school. There was an outside provocation, with the NCETM materials, and Alf’s presence, but the process was about the teachers and leadership in the school developing their practice without a pre-conceived notion of what this development would lead to. We see a parallel between the way teacher J, in the lesson, was genuinely interested in the thinking of his students and the way that the teachers in discussion seemed to be openly engaging in each others’ thinking. For teacher J, a focus on “methods not answers” supported his attention to student L’s thinking. In a direct parallel, in the use of video, Alf has an initial focus on “observations not judgments”, without an agenda about the particular things teachers will notice. We conjecture that a “meta” focus of a teacher or facilitator supports participants in making their own sense and seeing the purpose of tasks as relevant to their own thinking.

Finally, we are convinced that the commitment of the headteacher was absolutely vital. The model came from the sustained interest from the head in getting teachers opening up their classrooms to each other and in learning from each other. It is with great sadness we report that Sue East, the headteacher at the time, passed away in December 2018. She was inspirational in her commitment to others and her belief in the creativity of students and staff alike. She is sorely missed.

Alf Coles is an Associate Professor of Mathematics Education at the University of Bristol.

Katherine Evans was working at St Andrew’s Church School, Bath as an early years teacher at the time of this project, as well as working alongside Alf as a member of the research team. She now lectures in Early Childhood Studies at the University of Plymouth.

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Six years ago, probably the only time mathematics teachers in England came across the word “master” in their professional lives was if they got entangled with a master page on presentation software. Roll forward to 2019, engaging with “master” and “mastery” in mathematics education is unavoidable. This will continue for at least the next four years, given the government’s continued commitment to funding their mastery policy until 2023, as the vehicle for promoting East Asian teaching methods in mathematics. The policy is premised on the hope that results in England will rise to be more like those found in East Asian jurisdictions in international comparative tests like The Programme for International Student Assessment (PISA) study of 15-year-olds’ performance in mathematics, science and reading.

Another place mathematics teachers may have come across the idea of mastering, is when buying music. The digitisation of music created opportunities to remaster original recordings of songs. New technology allowed enhancement of the musical quality of old tapes, though whether removing the signature tones of vintage technology is an improvement might be disputed by some music lovers. A remastered version would be close to the original but not simply a direct copy. Digital music also opened up possibilities for the flowering of the remix culture, particularly in dance and electronic genres. Similar to remastered versions of older songs, some remixes of new tracks are close to the original recording, perhaps longer or with particular parts extended, or with some of the vocals taken out. Other remixes can be dramatically different, perhaps introducing the style of a different sub-genre, so that whilst the kinship between the original and the remix is recognisable, the remix sounds like an almost different track. To further complicate the process of distinguishing between different tunes, sometimes producers would add in samples from other tracks into their remixes. Taking sampling a step further a new genre has emerged of “mash-ups”. In a mash up, two or more tracks, or parts of them, are overlaid. When mash-ups work well they offer creative, and often ironic, juxtapositions where the sum is more than the parts. But when they do not work, or are done badly, there can be a lot more ‘mash’ than ‘up’.

My aim in this article is to consider the mastery policy as a whole, to look at where we have come from, where we are now and to consider where we might be going. One way of thinking about mastery, in its various sub-genres, is whether they are remasters, remixes or mash-ups and what are the sources for these reworkings. My perspective is informed by witnessing the policy develop up close through the evaluation of the government’s flagship mathematics teacher exchange with Shanghai.

The evolution of mastery in mathematics education in England

Regardless of the policy commitment, mastery is a slippery word that can and is used in lots of different ways. There are many different actors that promote practices, policies and resources under the mastery banner that have overlaps but are distinct, with roots in teaching traditions in both Singapore and Shanghai. Given the differences between programmes and resources, it may be better to refer to masteries than mastery.

Mastery is listed in the Sutton Trust toolkit (Higgins et al. 2014) as being an effective approach to teaching mathematics; this predates the current use of the word “mastery”. The evidence cited refers to Bloom’s “mastery learning”, which refers to cycles of teaching, learning and assessment. Bloom developed his ideas in the 1960s, introducing what was then a radical idea that if material had not been learnt then it could and should be represented using different materials or examples, and if this process was repeated then learning would take place. Arguably, many of Bloom’s principles have been adopted and integrated into mathematics teaching and have become the norm, though in England there are advocates of a return to a more purist version of Bloom’s original approach.

Bloom was a US scholar. However, today the term “mastery” is associated more with East Asian mathematics. East Asian influences are not new. East Asian practices directly influenced the National Numeracy Strategy with an emphasis on mental
In 2007, the *Maths - no problem!* translation of Singapore textbooks were produced. In around 2009, a multi-academy trust, Ark, also looked to Singapore for ideas as well as other education systems that were deemed to be high performing. However, the mastery approach that was developed is described as being rooted in developments in England (see Drury 2014). Ark mathematics was renamed *Mathematics mastery* around 2010 and was supported by Education Endowment Foundation funding. This appears to be the first time the term mastery was used in relation to East Asian mathematics, at least in England. Although government attention has more recently switched to Shanghai, there has been an organic influence of aspects of Singapore practice in England, partly due to ideas found in *Maths - no problem* and *Mathematics mastery*. One aspect of this has been an emphasis on a problem-solving orientation. Another has been the idea of the Cognitive-Pictorial-Abstract (CPA) heuristic that has become something of a meme in primary mathematics in England. The CPA formulation is a reworking of Bruner’s enactive-iconic-symbolic distinction that has previously been advocated by various mathematics educators based in England (see *What is the P … in CPA* in MT265).

To use the musical metaphors introduced earlier, *Maths - no problem* appears to be either a successful remastering or remix of Singapore mathematics. The ascription of “successful” here is that it has been successfully implemented by teachers in English contexts and this has supported professional learning, rather than evidence, thus far, in terms of success in raising learner outcomes in comparison with alternative approaches. *Maths mastery* appears to be more a remix and development of previous successful English practice that was refined through consideration of similarities and differences with international practices, rather than a simple reformatting of Singapore practice. However, Singapore mathematics education itself can viewed as something of a mash-up, not only through the influence of Bruner, but also Skemp’s relational and instrumental understanding, and in the mix is both the East Asian (for example, Japanese) and Western emphases on mathematical problem solving. The English Cockcroft report (see MT243) has also been cited as an influence on developments of Singaporean mathematics education.

In 2012 and 2013 two study visits were funded by the English Department for Education to Shanghai. The first teacher exchange took place with Shanghai in 2014/15 and this has been at the centre of government mathematics education policy since. Why Shanghai? One answer can be found with the growing influence of PISA test outcomes on policymakers internationally. At the time of the exchange, Shanghai topped the PISA table in mathematics, though later it was replaced by Singapore. However, the teacher exchange was related to wider trade negotiations between England and China, that have been ongoing. In 1997, it was notable and surprising that, the then Prime Minister, Tony Blair introduced the first National Numeracy Training video. At that time, the political interest in mathematics teaching was for domestic consumption. Now, the importation of Shanghai mathematics teaching appears caught up in international trade deals. Government re-interpretation of recent evaluation findings of the Shanghai exchange, which did not find compelling evidence that the importation of Shanghai practice will lead to improvement, appears as much aimed at an international as domestic audience.

By 2014, the National Centre for Excellence in Teaching Mathematics (NCETM) had adopted the word “mastery” and was writing about mastery approaches. From this, an approach to teaching, *Teaching for mastery* (TfM), has developed, partly influenced by what schools involved in the original Shanghai teacher exchange did in practice following the exchange experience (see NCETM 2016). The TfM programme consists of a professional development programme to train mastery specialists, support for the specialists to work with groups of teachers from local schools (called mastery advocates), a subsidy to buy textbooks, as well as further exchanges with Shanghai teachers. The aim is for over half of English primary schools to engage with the TfM programme in some way, and this approach is also advocated for secondary schools. Although the funding for the TfM programme is a small fraction of what was available for schools when the National Numeracy Strategy was introduced, it makes up nearly all of the directly government funded support for mathematics curriculum and teacher professional development.

The TfM programme is easier to summarise than the TIM teaching approach. Indeed, the NCETM have stated that it is not easily describable. There are aspects of TfM that are recognisable as being derived...
from Shanghai practice, arguably remixes. There are other aspects of TfM, or at last ways these are described, which appear more derived from previous practices in England. However, their origins are not always clear. To return to the music metaphor, TfM appears to be something of a mash up, combining and overlaying different ideas about teaching mathematics. This happens at the policy level of the advocated approach. However, in our evaluation, we also found that a similar process happened following the Shanghai exchange at the school level with lots of reinterpretations of both Shanghai practices and NCETM TfM advice. Schools and teachers combined and adapted what they had seen and understood as Shanghai practice, as well as the TfM principles the NCETM promoted, with what they were doing locally and, in many cases, with other ideas such as those from Singapore. Being a mash up is not necessarily a bad thing, but it is misleading when people refer to TfM as “Shanghai maths”.

Although the NCETM is the key player in implementing government policy, other actors are important. Further translations of Singapore mathematics textbooks and Shanghai sourced materials have been published, and various other professional development opportunities badged under the mastery label are on offer. The NCETM is publishing a series of “professional development (PD) materials” that appear to look a lot like curriculum and teaching resources. Perhaps they are labelled as PD materials to avoid the danger of being seen as competition to the text books that the NCETM approve as being worthy of government subsidy. Another influential set of resources is produced by White Rose Maths. White Rose started out as part of the Maths Hub initiative and is now an independent player in the market, also offering mastery and more general mathematics teacher professional development. To add to the potential confusion for teachers, shortly a text will be published with the title Teaching for mastery (McCourt, 2019) that differs in important respects to the NCETM’s advocated approach.

Given the different offers, it is not surprising that both mathematics teachers and school leaders express confusion as to what mastery in mathematics means. Partly, this is the outcome of a mirage of marketisation in which the government appears to be hands-off from central direction, thus creating spaces for, and to an extent encouraging, a variety of mastery initiatives and commercial offers. However, the government retains a strong influence on what is promoted to teachers and schools, even down to approval of detailed formulation of what will and will not be supported by government resources for materials or professional development.

**Mastery: What’s the evidence?**

The justification for looking to East Asia to improve attainment outcomes is based on East Asian countries comparative success in international tests. However, behind the policy is the assumption that this success is due to teaching methods, which the government aims to import, rather than cultural and systemic factors, which the government does not intend to import and appears to ignore. As much as we might like to think that teachers and teaching are the most important aspects of children’s success in mathematics, some of the biggest predictors of outcomes lie outside the classroom. Similarly, ways of teaching cannot be easily separated from teachers’ conditions of service. So whilst the government is keen for teachers in England to adopt Shanghai practices they do not provide the amount of time for preparation and professional development that may be needed to teach in the way Shanghai teachers do.

It should not be surprising that evaluations of the impact of adopting mastery teaching methods in England do not indicate that the politicians’ hopes are likely to be realised, given that some of the most important influences on attainment are not replicated. Randomised controlled trials of Mathematics mastery suggest potentially small benefits and these are more likely at Key Stage 1. More recently, evaluation of the outcomes of schools who engaged in the first Mathematics Teacher Exchange (MTE) with Shanghai in 2014/15 and who then implemented mastery practices similar or the same as those advocated by the NCETM, found no effect at KS2, and again a small effect at KS1 (Boylan, et al. 2019). There is evidence, however, that some of the component practices of mastery may improve attainment. These components include greater interactive dialogue and greater use of representations (see Boylan and Rycroft-Smith, 2019).

Putting aside the important issue of effects on attainment, there is much that is positive in the mastery innovation. Teachers engaging with Shanghai methods have also been encouraged to develop and have experienced the collaborative forms of professional development that are routine in Shanghai. Many teachers who engage with mastery report positive professional development outcomes. It may be that the findings of improvements in Key
Stage 1 outcomes are as much to do with increased teacher subject knowledge and confidence as with particular teaching methods. In addition, the mastery innovation has opened the space for discussion by classroom teachers of important ideas and theories about learning mathematics such as variation theory.

Looking ahead

Given that there is as yet little evidence to support the current policy in terms of attainment outcomes, there is a need to investigate in more depth which of the different practices that are put forward under the mastery banner are most effective, as well as what other, possibly unintended, consequences there might be. There are a host of important questions that the mastery innovation opens up. Some of these require large-scale research, but others require local answers gained through a range of teacher activity from daily professional experimentation to more formalised action research. For example, one of the promoted features in some versions of mastery, including the NCETM’s TIM, is the idea of slowing down the pace at which content is covered in order to teach in more depth. This is summarised as teaching “small steps”. This is open to many interpretations, some of which risk fragmenting the mathematics curriculum into bite-sized, disconnected pieces. Alternatively, other interpretations of “small steps” invite looking in depth at mathematical concepts such as zero or equality that might previously have been skimmed over in a rush to demonstrate pace and progress in a single lesson. However, simply exhorting teachers to slow down does not give a sense of what speed is suitable and for whom, when and for what areas of mathematics. Also given many of us have the experience in our mathematical histories of ourselves or others making “big leaps”, the relationship between steps and leaps is worthy of investigation.

Similarly the slogan of “intelligent practice” promoted by the NCETM appears self-evidently a good idea. Unintelligent practice would not be desirable. But a desire for students to practise intelligently does not, in itself, develop a sensitivity to the relationship between procedural fluency, conceptual understanding and learner self-reliance found in approaches that seek to develop the mathematician as well as the mathematics.

Currently, I believe there is an unhelpful tone of certainty with some who are putting forward mastery as a solution to perceived concerns in English mathematics education. Perhaps, this comes because the mastery market place appears increasingly crowded and so the surety is similar to a sales pitch. Perhaps because for some adopting East Asian teaching approaches represents something of a conversion in their approach to teaching, this leads to a tone of zealotry or evangelism. Perhaps for others, this certainty comes because mastery is being seized on as a vehicle to promote beliefs and views long held, but ones that have not hitherto had much space for a hearing. Whatever the reason, whilst certainty is appealing to policy makers and politicians, the engagement with East Asian mathematics should serve to highlight that there is much we do not know about East Asian practices, those that have pertained in England previously, and what from East Asia is applicable in our context.

What is needed is a careful sifting of evidence and a willingness to ask challenging questions. For example, in the selective filtering of Shanghai practices, it is often overlooked that children in Shanghai do not start formal schooling until 6 years old. Thus the notion of Shanghai teaching methods applied to reception or year 1 is problematic. One example is the general advice by the NCETM for children to routinely be sat in rows for mathematics without qualification. Such advice runs counter to usual practice in England for reception and year 1 (as well as older children in many schools) of carpet time, the environment in which whole-class teaching would generally take place. In this regard, the evidence for positive effect at KS1 should not be read as an endorsement of the NCETM seating policy, given that in both Mathematics mastery and most Mathematics Teacher Exchange schools this seating approach was not applied at KS1. In general, in adapting methods developed for children that are 6 years and younger, a degree of humility and willingness to listen to Early Years specialists is needed. If this does not happen, the mastery agenda risks adding to the ‘schoolification’ of the Early Years. It may be that ‘deschooling’ Year 1 may serve children better in the long run even if attainment is the driving motivation, rather than other values and purposes for early primary education.

Currently, the ATM is engaged in discussions with sister organisations about developing a single subject association. The current systemic picture outlined above highlighted two trends, increased political direction or at least influence on how mathematics should be taught but in tension within an environment of increased competition, and paradoxically some
of this competition coming from state-funded or subsidised actors. In this environment, the role and voice of a strong independent subject association will be vital to: Support teachers to navigate an increasingly complex and confusing environment; to ask uncomfortable questions of those in positions of power; and to encourage teacher-led enquiry into the host of interesting issues and questions that the mastery innovation draws attention to. To further stretch the music metaphor, the provenance of remasters, remixes and mash ups is less important than mathematics teachers being skilled DJs who can select and blend different tunes as appropriate to the needs of the situation.


Mark Boylan is a Professor of Education at Sheffield Hallam University.

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COMING SOON
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www.atm.org.uk/store
As someone involved in primary initial teacher education, I would like to present a slightly different perspective on the current mathematics mastery debate. This is one which I have not seen expressed elsewhere, but which to me seems significant in informing the discussions we might have around this approach to teaching mathematics. I would like to step back, take the long view, and look at the mathematics mastery approach within the context of the history of primary mathematics education over the past 50 years.

It would appear that a mathematics mastery approach is seen by some currently as the key way to develop mathematics teaching and learning in England, with, for example, considerable funding being provided by the government to promote the approach through the Maths Hubs programme run by the National Council for Excellence in Teaching Mathematics (NCETM). However, I have also heard many involved in primary mathematics education express the view that much of what is being proposed is nothing new. Recently however, two comments, one heard during a Maths Hubs CPD session and one read on a mastery website from a different organisation, prompted me to think about this whole debate slightly differently. It is these thoughts I share with you here. One of these comments was that, whereas Shanghai had taken on board the messages of Cockcroft (see Cockcroft, 1982), we in England had not. The second, that, “until recently, teaching children to follow a process had been at the core of mathematics education” ... but that ... “with the 2014 curriculum, this began to change” (see https://thirdspacelearning.com/resource-maths-mastery-ultimate-guide/).

From my understanding of the development of primary mathematics education in England since the mid-twentieth century, I would argue that both of these comments misrepresent the situation here. During that period, primary mathematics education in England was following what has been referred to as a “reform agenda”, with a move away from transmissive teaching to a focus on procedure or following a process. Instead, and as espoused by the Cockcroft Report, came greater focus on the development of conceptual understanding through, among other things, practical work and the use of a range of resources, as well as focus on the use and application of mathematics alongside a recognition of problem solving as a key element of mathematics education.

During this time, many of the elements often associated with a mastery approach were present, in the mathematics education literature and debate and practically in the classroom, albeit not combined in this specific format and with this particular label. For example, many argued that mathematical teaching and learning needed to focus on conceptual understanding, and not just procedures, facts and formulae and that this was developed by, among other things, making connections between different areas of mathematics and the use and application of mathematics in solving problems (see Richard Skemp in MT77). In the last MT, Anne Watson and John Mason explained how the work of Bruner on enactive, iconic and symbolic representation has underpinned the use of resources and images within mathematics teaching and now emerges in the concrete-pictorial-abstract approach.

So, maybe the “heard it all before” camp do have a point? I would argue that, yes, to a certain extent they do, but I believe there are more important considerations to explore, which become apparent if we continue with the long view of mathematics education. Despite the “reform agenda”, the current government clearly believe there is still an issue with mathematics teaching and learning that needs tackling. This concern appears partly to be informed by the UK’s position in the PISA rankings, but it is important to note that those elsewhere in the field of mathematics education have also voiced concerns. Leone Burton noted that there was evidence that much mathematics teaching continued to be transmissive (see Burton, 2004) and the Williams Report highlighted the need for increased focus on the use and application of mathematics and problem solving (see Williams, 2008).

In which case, if we are taking the long view, I would suggest that, rather than deciding whether we sit in...
the mastery camp, or in the “heard it all before” camp, it might be more useful to consider a different set of questions. If the various aspects of mathematics teaching which we see combined within a mastery approach have been identified for some time as effective mathematics teaching, even if not in this specific combination and format, and are often not actually being put into practice in the classroom, a key question would seem to be, “Why not?”. If we want to improve primary mathematics teaching, whether through a mastery approach or otherwise, comes the question, “What can we do to make it more likely that these approaches will be adopted in the future?” Perhaps this is common ground for all in the mathematics education community. However, without some understanding of what has gone before, the importance of even asking these questions may not be recognised.

To my mind, answering these questions involves discussions around the curriculum itself. For example, one aspect of a mastery approach is to cover fewer topics in more depth. It also involves debate on the current assessment system and the impact this has on primary mathematics teaching. There are questions around access to resources and teachers’ own mathematical subject knowledge. The teachers who teach mathematics in primary schools in England are usually not mathematics specialists, unlike in Shanghai. Perhaps CPD for teachers and other adults in the classroom is the key?

A mastery approach may be the buzzword of the moment, but it sits within the broader flow that is the development of mathematics education over time. As such, it seems reasonable to suggest that anyone seeking to promote this as an approach is likely to gain from glancing back over their shoulder and reflecting on what has gone before. Taking the long view also shows us that improving mathematics teaching and learning is an ongoing challenge. Recognising what has been tried previously, what succeeded or failed and reflecting on why, may help address those challenges in the future. My own opinion is that no one approach is going to provide the answer, but that progress comes from a willingness to be open minded, to engage in debate and not to get stuck too firmly in any one camp.

Fiona Tidbury lectures in primary mathematics education at the University of Cumbria.

References
Unlike the previous two pieces, this is a description of some teaching that preceded the mathematics club meeting. I felt the need to tighten some loose ends in our emerging knot-theory notation. The first two meetings (see parts 1 and 2 in MT264 and MT265) were about exploring knots and ways to describe them using notations and we made some impressive strides. But at this stage, I felt the need to steer the children in the right direction. So, I decided:

- We are going to use \( I \) rather than 0 (zero) to denote the identity of composition. Even though 0 is a neutral element for addition, as correctly spotted by the children, using it as the neutral element for composition is not idiomatic. \( I \) or 1 (one) are more common, for various reason I will not get into here.

- Using a fraction-like notation for “parallel” (tensorial) composition is also quite unidiomatic, although it is intuitive. I shall introduce the standard \( \otimes \) notation instead.

- We will stick with a generic duality operation \( _* \) so that our “unit” and “co-unit” are \( C \) and \( C^* \) rather than \( C \) and \( D \) as some of the children insisted on.

These are of course small matters which we could put up with, at the expense of becoming incomprehensible to the rest of the world. The more important thing, for me, is that the interplay between sequential (functional) composition and parallel (tensorial) composition is not very easy to get right. Look at our overhand knot (see figure 1):

The individual components are well identified but it is unclear how to put them together. There is a sequential composition at the top \( (X \cdot X \cdot X) \) but how to connect that with the \( C, I \) and \( C^* \) underneath is not clear from the decomposition. \( X \) interacts with \( C \) and \( C \) with \( I \) but not obviously sequentially or in parallel.

The way out is to introduce the concept of ‘types’. We can give knots a type \( (m, n) \in \mathbb{N}^2 \). The left projection represents how many “loose ends” stick out to the left and the right projection how many to the right. Types tell us what can correctly compose with what. So, this is a \( (4, 6) \)-knot:

Note the implicit restriction: no loose ends are allowed to stick out from the top or from the bottom, because if a bit of string goes down or up we can always bend it left or right then extend it until it lines up nicely with the other loose ends. I hoped this topological invariance of knots will be intuitively clear to my young learners.

So, the types of our basic knot parts are (see fig. 3):

These are of course small matters which we could put up with, at the expense of becoming incomprehensible to the rest of the world. The more important thing, for me, is that the interplay between sequential (functional) composition and parallel (tensorial) composition is not very easy to get right. Look at our overhand knot (see figure 1):

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So, the types of our basic knot parts are (see fig. 3):
Now we can say that we only allow the composition $K \circ K'$ for knots $K: (m, n)$ and $K': (m', n')$ if $n = m'$, resulting in a knot $K \circ K': (m, n')$. Here are two composable knots (see fig. 4):

Figure 4: Two composable knots.

And this is their composition:

Figure 5: Knot composition.

Note that because these are strings the loose ends do not need to be geometrically aligned. It is enough to have the right number of loose ends to be able to glue them together. And this is their actual composition (see fig. 6):

Figure 6: The actual composition.

Any two knots $K: (m, n)$ and $K': (m', n')$ can be composed in parallel in a knot $K \otimes K': (m + m', n + n')$. The parallel composition of the two knots above is (see fig. 7):

Figure 7: Composition of knots 1 and 2.

So now everything is in place to describe our overhand knot algebraically (see fig. 8):

Figure 8: The algebraic notation.

The overhand knot is therefore described by the formula

$$(I \otimes C) \circ (3 \cdot (X \otimes I)) \circ (I \otimes C^*).$$

As a simple notational improvement we can introduce $\cdot$, a ‘scalar’ product, shortcut for repeated sequential composition, leading to the more concise

$$(I \otimes C) \cdot (3 \cdot (X \otimes I)) \cdot (I \otimes C^*).$$

The identity string $I$ is needed, not just on the bottom, but also to the left and to the right, in order to make the loose ends match properly in sequential composition. This kind of padding with identities will be a useful trick that will allow us to define a lot of knots compositionally. Next issue, I return to our young learners.

Dan Ghica is a reader in the semantics of programming languages at the University of Birmingham.

This article first appeared as a blog in 2015. The club was held at Water Mill Primary School, Birmingham.
Liverpool counts: Using mathematics to transform the lives of young learners

Tony Cotton and Jackie Salim-Smith discuss the Liverpool Counts Quality mark: An intervention implemented in schools across Liverpool.

In many primary schools in Liverpool, the children who come to school suffer multiple deprivations, with many of the children living in poverty. The Child Poverty Action Group (CPAG, see www.cpag.org.uk) define children from families who struggle to pay for essentials and to participate in society as “living in poverty”. Despite these children having little in material terms, several schools have set up projects with parents and children to develop financial literacy, or should that be financial numeracy, with the children.

These projects include working in partnership with credit unions run, on a day-to-day basis by the school. Parents use it to save for special occasions and for emergency borrowing, allowing them to avoid the emergency loan companies, who penalise poverty through exorbitant interest rates. It also offers the opportunity for parents to understand how to better manage their meagre resources. Alongside this the children learn about saving. They bring 10p to school every Friday and choose whether to spend immediately or to save. Each classroom has a shop with items at 10p, 20p, 50p and so on. So, children become tempted to save for a few weeks to buy something from the 50p shelf.

This may be one facet of activity in schools that have been awarded the Liverpool Counts Quality Mark, an initiative developed in partnership between School Improvement Liverpool Ltd. (SIL) and the Liverpool Learning Partnership (LLP). The priorities for Liverpool Counts include:

- to raise standards in mathematics and numeracy with fun and joy of number at the heart of all aspects of the strategy.
- to challenge perceptions and change attitudes towards number and mathematics.
- to raise the profile of numeracy through a range of activities, events and resources with an emphasis on the application to real life contexts.
- to bring relevance to numeracy through problem solving set in local contexts.
- to involve parents and the local business and cultural communities.
- to create a sustainable legacy of a love of numbers.

In his forward to MT263, a special issue exploring Mathematics and the living world, Richard Barwell asked what a fully integrated approach to teaching mathematics for the living world might look like. I wonder if this approach might be an exemplar?

A first step for schools wishing to be awarded the quality mark is to explore and develop the culture and ethos around mathematics in school. All schools are encouraged to, not only develop positive attitudes towards mathematics, but, actively discourage negative attitudes from staff, teaching assistants and parents or carers. Teachers are asked to ensure that they:

- are aware of what they say about mathematics, especially around children.
- challenge anyone they hear making negative comments about mathematics.
- discuss the issues with pupils whenever they hear celebrities in the media saying that they “cannot do mathematics” or being negative about mathematics.
- share their own enjoyment of mathematics and highlight when they have used it in everyday life.
- dispel the myth that there is a mathematics gene and that only a few people can be good at mathematics.
- encourage parents to be positive about mathematics.

To receive the quality mark, schools collect evidence to show that they meet criteria under the following six themes:

Leadership and management: The expectation here is that there is an agreed approach to the promotion and development of mathematics for life and that this is evident in school-planning documentation.

Workforce development: Schools are expected to nominate a mathematics champion who leads on ensuring all staff are engaged in supporting the aims of Liverpool Counts. This includes ensuring that there is evidence of mathematics being taught through other curriculum areas.

Promotion and development of mathematics: A school
awarded the quality mark will be able to show that pupils regular engage in problem solving activity with a focus on solving real-life problems. Some links have also been made to the philosophy for children (P4C) approach, (see Rod Cunninghams’s article in MT263). In one quality mark school pupils were asked to discuss the question, “Is taxation fair?”

**Numeracy events and groups:** Schools are expected to make links to local, national and international events. One school holds a “daily mile” event where pupils are encouraged to walk a mile around the playground everyday. Schools have linked events such as the Macmillan big coffee morning to mathematics in the school and recently many schools used the context of the World Cup to develop mathematical activities.

**School-wide opportunities for mathematics for life:** A Liverpool Counts school will have dedicated areas and displays around the school to encourage both problem solving and making links between mathematics and the lived experience of the pupils and their families. This may be through reports of talks by parents or community members who have described how they use mathematics in their jobs or through a display that challenges pupils to solve puzzles or engage in mathematical investigations.

**Family and community involvement:** As well as encouraging parents, carers and members of the community to share their mathematical experiences with the pupils, Liverpool Counts schools will offer regular sessions to support parents and carers in developing their own mathematical skills.

It is the mathematics champions who lead on these areas supported by mathematics buddies who are appointed or selected from the pupils in the school. These buddies have supported teachers in running mathematics breakfasts at which mathematical puzzles and problems are enjoyed over breakfast before the start of school; running lunchtime mathematics clubs; engaging in peer tutoring of mathematics; running tuckshops or even creating a series of videos which explain to parents and carers how the pupils might approach calculations.

Tony and Jackie were invited into Smithdown Primary school to see this approach in action. Smithdown describes itself as an open and caring community and the school website suggests that, “The multicultural nature of the school population and its inner-city locality, close to the heart of Liverpool are a constant source of inspiration for children’s work and activities.” This is supported by a recent inspection report which agrees that there is, “a highly inclusive learning environment in which everyone is valued and diversity is celebrated. … pupils, who between them speak 43 different languages in addition to English, feel warmly welcomed and secure in school.” The inspector suggested that the approach to mathematics supported by Liverpool Counts and championed at the school had played a part in making sure that all pupils make progress in mathematics, indeed, the pupils progress exceeds that of all pupils nationally. But what does this look like? The mathematics buddies at Smithdown are busy people. There are two buddies in each year group who work closely with the teacher who has taken on the role of mathematics champion. Activities that have taken place across the school include “The fiver challenge” in which each class was given £5 and had to use this investment as a resource to create more money for a charity. This activity raised £320 and awareness of budgeting and profit. The school has organised “Debt aware” workshops for parents and the buddies organise a school tuckshop. The proceeds from the tuckshop have been used to buy library books with a mathematical focus. When we visited we were shown the development of an edible playground which was being constructed in order to raise awareness of healthy eating. There was palpable excitement about the mathematical possibilities here.

The Liverpool Counts assessment document describes how the pupils speak with “clarity and consistency about their love of maths” and of how they now see mathematics everywhere, in school, at home and on the journey between. These young mathematics champions have also taken on the responsibility of teaching their parents to understand how to use mathematics in their everyday lives and to grow to love mathematics. The school has built on this by running several workshops for parents which has led to OCR accreditation for parents taking part.

To return to Richard Barwell’s challenge in MT263 and to respond to the questions he posed at the end of his introduction:

Is a fully integrated approach to developing awarenesses of mathematics for the living world desirable? Yes.

What might it look like? The above.

Might it be dangerous? No.

Might it lead to a challenge to the current system? Yes.

But, with Richard Barwell, and the young people of Liverpool, we would argue, “This is precisely what is needed.”

**Tony Cotton is editor of Mathematics Teaching.**  
**Jackie Salim-Smith is a member of School Improvement Liverpool and the Mathematics Teaching editorial board.**
The 2019 ATM/MA conference was a big success. Held at Chesford Grange, Warwick from 15 to 18 April, the conference facilitated four days of incredible continuing professional development for all of us who are passionate about mathematics and its teaching and learning. With over 80 sessions to choose from and over 260 delegates to network with, this year’s conference was one of our biggest to date. The conference was very well received particularly among those who were new to our community as one delegate tweeted: “First ATM/MA conference for me and it was just excellent - learnt so much and really enjoyed it too. Wonderful the kind of energy and thinking a group of great maths teachers can create”.

The plenary speakers were a hit. In his Opening Plenary titled ‘Listen to the learners - learn from them’, Colin Foster argued how students’ mathematical comments and questions can and should be used by practitioners to develop their own insights about mathematics and mathematics pedagogy. In her Closing Plenary titled ‘Between the secret garden and the hothouse’, Sue Gifford (an early years mathematics expert) gave several interesting examples to highlight how young children’s enjoyment of humour and efforts to make learning activities playful could be utilized by practitioners to make early years mathematics teaching and learning more effective and enjoyable.

At the Annual General Meeting, Professor Anne Watson was announced as this year’s recipient of the ATM Life Membership for her years of contribution to our community. Louise Hoskyns-Staples was elected as Honorary Secretary, succeeding Corinne Angier whose term has ended, and we thank Corinne for her tireless efforts over the last three years. A range of important motions were also put forward including a proposed two-year trial of free student e-membership and two motions concerning discussions about the proposed single mathematics subject association.

Social events were also among some of the conference’s highlights. The ATM-sponsored Newcomer/Student/NQT Introduction & Drinks event, led by Bob Vertes, saw 40 new members of our community learning what ATM represents and why they should consider becoming a life-long supporter of this association. The quiz hosted by Colin Penfold was as popular as ever. The ATM Workshop, hosted by Jayne Stansfield, was busy throughout the conference. (The fact that the Workshop was located in the bar area presumably helped a bit!) The Musical/Open Mic event, the Ceilidh Band and disco all gave our conference delegates a chance to let their hair down. We also thank our friends at NRICH for running a very enjoyable special session, resulting in some highly creative marble runs!

This conference would not have been as successful as it was without the incredible ATM office team who worked tirelessly behind the scenes. The team includes Sam Walters (our new Executive Officer), Karen Foster and Laura Bandell. Special thanks go to Karen Kirkley (our Conference Officer) whose experience in organising the past 18 ATM conferences ensured our delegates had a really positive experience at the conference.

We are also grateful for the generous support of our Headline Sponsor (White Rose Maths), Major Sponsor (PG Online) and all our exhibitors. We also thank Anne Haworth who kindly sponsored the Wine Reception.

Looking ahead to 2020, the General Council and the office team very much look forward to welcoming you to our next annual conference, which will be held at the four-star Staverton Park Hotel in Daventry, Northamptonshire from 6 to 9 April 2020. The conference’s theme is ‘2020 Visualising’. More details of the conference can be found on the ATM’s website.
Thinking mathematically everyday

These activities are taken from *Thinking mathematically everyday – Y3: Number, geometry and statistics*.

Introduction

This booklet is for Year 3 children and three tasks have been chosen, on Number, Geometry and Statistics, to show how they can work in a Year 3 classroom. They are linked to the programme of study (POS) for the year group.

There are ideas for support and extension as well as images to help the children think about the tasks. These tasks show that there is no need to teach fluency, reasoning and problem solving separately from the rest of the curriculum. Content can be introduced, developed and practised through learners working on these tasks. Having seen how that can work with these particular tasks you can use the generic questions and prompts when learners are working on other tasks. These may be from other ATM publications or websites such as NRICH.

The ideas and prompts are given for promoting conceptual understanding and these are separate from those for reasoning and those for problem solving.

You can decide whether you want to focus on developing understanding, reasoning or problem solving at that particular moment. However, whichever you choose it will also develop the other two – an active brain, making sense of mathematics, will naturally progress in all three aspects.

This booklet is one of a series for each of years one to six to inspire teachers to embed reasoning, problem solving and conceptual fluency in all of their lessons.

1 – FINDING FRACTIONS

POS statement(s)

- Y3 recognise, find and write fractions of a discrete set of objects: unit fractions and non-unit fractions with small denominators.
- Y3 recognise and show, using diagrams, equivalent fractions with small denominators.
- Y3 compare and order unit fractions, and fractions with the same denominators.
- Y3 add and subtract fractions with the same denominator within one whole: (for example, \( \frac{5}{7} + \frac{1}{7} = \frac{6}{7} \)).
- Y3 solve problems that involve all of the above.

Key prior learning

- Y2 recognise, find, name and write fractions \( \frac{1}{3}, \frac{1}{4}, \frac{2}{4} \) and \( \frac{3}{4} \) of a length, shape, set of objects or quantity.
- Y2 recognise the equivalence of \( \frac{2}{4} \) and \( \frac{1}{2} \).
- Y2 write simple fractions, for example, \( \frac{1}{2} \) of 6 = 3.

*Thinking mathematically everyday – Y3: Number, geometry and statistics* is available from www.atm.org.uk.
Making connections

- Connections to dividing numbers (sharing): \( \frac{1}{2} \) of 12 = \( \frac{1}{2} \times 12 = 12 \div 2 \).
- Factors and multiples.
- Number bonds (to 12).
- Connecting the word ‘of’ with multiplication (in upper KS2 when children are multiplying a fraction by another fraction).

Task – Finding fractions

Your challenge is to find different unit fractions of 12.

You have 12 counters.

Use them to find your fractions.

\( \frac{1}{12} \) of 12 = (or \( \frac{1}{12} \times 12 = \))

Now try finding different non-unit fractions of 12.

\( \frac{2}{12} \) of 12 = (or \( \frac{2}{12} \times 12 = \))

\( \frac{3}{12} \) of 12 = (or \( \frac{3}{12} \times 12 = \))

Which fractions give a whole number answer?

Record your fractions accurately on paper.

What do you notice about the denominators of these fractions?

Can you think of a question with the answer 1, 2, 3, 4, 5, 6, … 11, 12?

Are there different ways of getting to the same answer?
Ethics and the mathematics teacher

In the first of a two-part article, Paul Ernest asks, "How does ethics concern the mathematics teacher? What is ethical mathematics teaching?"

It seems clear that mathematics teaching is an ethical undertaking, for it is intended to educate students, to enhance their knowledge, skills and thus their life chances. Ethics is about the good, about behaving in a way that benefits others and enables their flourishing. Thus, ethics enters into all aspects of human life and professions and that includes the teaching of mathematics.

In my analysis, the ethics of the mathematics teacher can be seen in terms of two sets of nested responsibilities, first, those of all humans, and second, those of all professionals. The ethics of mathematics teachers is a special case of professional responsibility, and is treated third.

First, all human beings have responsibilities towards other humans and to society, as well as to the environment and the living world. Humans are social creatures who not only are and have been fully dependent on others but who are largely formed through their relationships with others. No one can become an adult, let alone a healthy and balanced one, without the care and support of others. We therefore owe everything to others, including being honest, respectful, caring, supportive and attentive to their needs (Levinas, 1972). This debt can be expressed in a number of ways. All religions promote the golden rule, “Treat others as you wish to be treated yourself” and the silver rule, “First, do no harm”, which is the Hippocratic Oath that medics swear. As well as having religious foundations, these rules have humanistic grounds, stemming from the social nature of humankind described above. They represent some of our universally shared human responsibilities.

In our responsibility to others there is no special class of persons that are included or excluded, or that deserve special treatment unless they are especially needy and require particular support, such as babies and children, the aged, the infirm and the disabled. Thus, the primary ethical responsibility to others, deriving from our humanity, commits us to equal treatment of and for all and thus to a socially just approach to others irrespective of social class, nationality, race, creed, religion, sex, sexual orientation, disability and so on.

Second, all professionals have responsibilities towards the institutions of which they are a part, and towards the roles that they undertake. Any professional, including the mathematics teacher, has ethical responsibilities to (1) support colleagues; (2) participate in supporting and enhancing the institution and its goals; (3) carry out their own professional duties to the best of their abilities; and (4) support and enhance their own profession and its standing in society, presuming this is warranted, as it normally is. Why do professionals have these responsibilities? To become a professional is to voluntarily accept a professional code of conduct and responsibilities in exchange for what is mostly pleasant and enhancing work, with protected job security, elevated social status, and reasonable financial rewards. Most vocational occupations are also personally enhancing for professionals because they work with a degree of autonomy in an environment of trust, and generally find fulfilment through deploying their capabilities, skills, and creativity in practice. In addition, professionals can take satisfaction from knowing that they are contributing to the overall good of society.

The responsibility to support colleagues can involve being a member of the appropriate professional associations or unions and participating in the training of younger colleagues. Supporting a professional institution and its goals may involve taking on senior administrative and managerial positions to help to sustain and enhance the institutions. Persons may participate and take on such positions for a variety of reasons and motives, including political motivations or personal ambitions, but providing they are working for the benefit of the institutions from some ethical perspective such involvement is ethically defensible, or in a word, good.

However, it should be acknowledged that there are ethical risks in taking on roles with power and privileges. First, there is the risk of becoming aligned with the institution at the cost of the interests of those represented and managed, if these diverge. As a leader in an institution one has the responsibility to represent the interests and well-being of one’s team and one’s clients, and to resist policies and practices antithetical to these interests, even if they come from on high. Second, positions of responsibility and power come with privileges and rewards. These are benefits associated with the position, enablers of the leadership role, and not personal entitlements.
of the role-holder. For leaders, there is the ever-present danger of succumbing to inflated notions of self-importance and entitlement. As the well known dictum says “power tends to corrupt” (and absolute power corrupts absolutely) (Dalberg 1887). Thus promoted roles of responsibility within an institution bring with them their own ethical challenges.

Third, a mathematics teacher has specific additional responsibilities because of the particular nature of their job of teaching mathematics to students. These are: (1) To treat students with care and respect; (2) to teach mathematics in an effective way that benefits the students; (3) to be engaged with the profession and keep up to date with research and developments, and, (4) to maintain their own interest and enthusiasm. Why does a mathematics teacher have these responsibilities? They follow from the responsibilities all professionals accept voluntarily in becoming a professional. That is, to carry out their professional duties to the best of their abilities, including respecting clients, practising their profession well, and enhancing their profession overall.

From the perspective of the mathematics teacher, this third set of responsibilities is the one that relates to the specifics of the job, that is teaching mathematics to students in a school or college.

However, before expanding on the details of these ethical demands, a caveat is needed concerning the high professional standards laid out here. It is a fact of life that these simple idealistic sets of responsibilities are frequently compromised and that this does not make the professionals themselves unethical. Such compromises may occur, first, because there are competing and conflicting demands within the context of professional practice. Second, normal human beings cannot operate optimally at all times. Such shortfalls are usually because of problems and conflicting demands within the personal life of the professional.

Within the professional situation, the school or college, in the cases I am considering, there can be a number of types of competing, conflicting and even contradictory demands. These can stem from many things including inconsistent or problematic management directives; disrespectful uses of power; complaints and challenges to professionals from insiders and outsiders (including inspectors, students and parents); inter-staff conflicts; staff shortages; unexpected disruptions including those cause by challenging students; resource shortages; overcrowding; environmental degradation; new curriculum and assessment demands and more generally changes in the professional situation that conflict with established practices. All of these can be accommodated professionally and ethically in a learning and growing institution that seeks to identify and overcome problems and obstacles. However, this requires the commitment and involvement of the leadership and managerial team in maintaining a values-driven ethos for the whole institution.

Secondly, personal life challenges may compromise professional functioning. Anything from illness, stress and family issues to financial problems and being a victim of crime may interfere with a professional’s ability to operate optimally. Provided that the individual has the active long term goal of reducing and overcoming these obstacles to effective professional practice, including seeking help where necessary, these are not significant ethical lapses. Optimal professional functioning should be a perpetual goal even if it is not always achieved or immediately achievable.

**Ethical mathematics teaching**

What constitutes ethical mathematics teaching is the most specific and unique aspect of the discussion of ethics and education from the point of view of the mathematics teacher. I distinguish three aspects.

First, there is the duty of care for one’s students, shared with all teachers. Second, there is the teaching mathematics effectively so as to benefit students. This is by far the most complex of these notions and responsibilities to unpick. Third, there is the engagement with the profession of teaching so as to keep up to date and maintain one’s enthusiasm. Engagement with the profession should also involve giving back: participating in professional bodies, reflecting on the nature of the mathematics curriculum and its assessment, maintaining up-to-date expertise and knowledge of relevant research, and supporting and contributing to the initial and in-service training of colleagues.

The adjective “effective” is troublesome because it hides a more complex relation. If we say an action is effective, we mean that the action is judged to be effective by a group of persons in attaining a particular set of objectives. Thus there are hidden dimensions concerning: Who makes the judgement? On what evidential basis? With respect to which objectives? There is nothing intrinsically mass-orientated, that is requiring a medium to large-sized class, in mathematics teaching or indeed teaching any subject. Teaching may be conducted by a teacher with varying numbers of students from a single one, to virtually any number, given suitable accommodation, planning and resources. There is also team teaching with multiple teachers or working with support teachers in a classroom, but neither of these brings in any completely new ethical responsibilities. Typically
teaching to groups of size 12, 30 or 60 is done in order to economise on teacher time and resources. There are of course benefits to whole-class teaching. Students can and should learn from peer interaction, and seeing other group members’ processes, strategies and errors displayed and discussed in class is a valuable teaching and learning technique that is difficult if not impossible to use in one-to-one tuition. On the other hand, there are benefits to individual or small group teaching. The teacher can devote a significant amount of time and attention to individual students to evaluate their responses to presentations of mathematics and tasks, to assess their progress, to attend to their working methods and come to understand their personal problem-solving strategies, to diagnose their strengths, weaknesses and needs, and to tailor an individualised learning/teaching experience to meet these needs. However, in suitably organised classes of 3 to 40 students a flexible teacher should be able to balance the benefits of whole group activities with individual attention, although sometimes with difficulty.

Attending to their individual needs is part of one’s duty of care for students. Treating them respectfully, benignly, equally and consistently is another part. This includes not singling students out for approbation or ridicule for lapses or errors in their mathematical reasoning, no matter how elementary, apparently stupid, or recurrent they are. There is strong if anecdotal evidence that being singled out and publicly criticised or humiliated for mathematical errors or lapses in class can lead to loss of mathematical confidence and even mathephobia or fear and hatred of school mathematics among sensitive students. One small negative interaction can have lasting deleterious effects. Likewise, one small positive interaction valuing a student’s insight or mathematical work can have lasting beneficial effects, impacting on the student’s attitudes to mathematical work and to mathematics in general. Neither of these outcomes can be predicted as they depend on students’ sensitivities, interpretations and emotional responses to varying stimuli in the moment. But, a teacher should always be sensitive to these possibilities.

The question of how one should attend to students’ individual needs in a whole-class situation leads to an important ethical dilemma. All classes contain students with a spread of achievement levels in mathematics. Should the teacher target the average achievement level in the class, choose teaching targets and learning activities accessible to all of the students, or focus especially on the highest attainers? One solution is to offer a range of tasks of different cognitive demands so that students work at the level that suits them best. One example of such tasks for a range of students are Rich mathematical activities that allow entry across a range of cognitive difficulty levels (see Griffin in MT212). However, although promising, there is as yet little published research on the proven efficacy of this approach. Overall, accommodating the various achievement levels of a class of students and setting appropriately demanding work is a significant ethical responsibility of the mathematics teacher.

However, a mathematics teacher should never lose sight of the fact that a student’s pattern of achievement is not a reliable reflection of their competence or ability. Various factors can depress a student’s achievement scores below the level of which they may be capable. So it is a vital ethical responsibility not to form stereotyped expectations of student abilities. The underestimation of the educational potential of female students in mathematics was for many years a factor that depressed their average achievement scores.

The responsibilities of teaching mathematics

The responsibility to teach mathematics in an effective way that benefits the students, is a complex and multifarious one. Value judgements are involved in deciding the effectiveness of a teaching approach, in a particular situation and deciding what is of benefit to the students. To determine the effectiveness of a pedagogical approach one needs some means of rigorously assessing its effects in terms of educational gains. Furthermore, such gains can only be established against a set of educational goals and objectives. Thus, to establish what benefits students one needs to have determined a background set of goals for their mathematical education. Ideally a set of aims and goals, properly determined, represents what is beneficial for the students and good for society, although it is conceivable that these two interests might clash. But there is no one set of goals good for all students, nor can a single set of goals be wholly beneficial for society, for it depends on values, priorities, as well as underlying ideologies.

Aims, curriculum and ethics

During the development of the British National curriculum in the late 1980s and 1990s I identified five interest groups contesting the aims and goals of the mathematics curriculum (Ernest 1991, see table 1).

Each of these five groups thought that their own aims were best for the country, for developing a good society, according to their own lights. However, it can be shown that such aims are not always best for all the students in school. To demonstrate this it
is necessary to evaluate each of the aims from an ethical perspective.

The first group, the Industrial trainers, have the main goals for the bulk of the populace of teaching basic mathematical skills and numeracy as well as a social training in obedience. This is to prepare a compliant workforce with the basic skills necessary for routine jobs. This group does not want education politised in order to prepare a demanding and non compliant workforce. These aims are not intended for the future elite who are educated in private schools and to which the National Curriculum does not apply. What is unethical about these aims is that they support an elitist stratified society that does not provide the best life chances for the masses. The good life of these workers, and the development of their knowledge, skills and interests beyond drudgery and material consumption is discounted and not supported. The goods of life are reserved for a minority elite at the cost of the masses.

The second group, the Technological pragmatists, have the aims of teaching the mass populace both basic skills and the higher knowledge and skills needed to solve practical problems with mathematics and information technology. These goals are industry and work centred, but they serve a meritocratic vision of society in which, through education, some persons from lower socio-economic backgrounds can become professionals thus having more rewarding careers both in terms of satisfaction and pay. The vision of society served is still an elitist and stratified one, but embodies permeable class barriers that allow for individuals to find their own level according to their educational achievements. This is a more egalitarian and ethical vision, but is superficial in considering only educational outputs (achievements) and not the inputs, namely the educational potentials of all students and what needs to be provided in order to realise their talents. Thus it ignores what Bourdieu terms “cultural capital”, the partly hidden cultural knowledge, material resources and enhanced attitudes that children from better off families usually carry with them into schooling to their own advantage.

The third group, the Old humanist mathematicians, have aims that are pure mathematics centred, trying to maximise student understanding and capability in advanced mathematics, including an appreciation of mathematics. This group have an elevated view of the intrinsic value of mathematics and believe it should be emphasised for all students, in so far as they are capable, to preserve the rigour of proof and purity of mathematics and develop more professional mathematicians. Mathematics is a good in itself, as well as being important and useful in society. But to distort the education of the masses to favour the less than 0.1% of the population who will become mathematicians and the less than 1% who will professionally apply mathematics is ethically unsupportable.

Each of these three groups strongly subscribes to a belief in inherited mathematical ability and is committed to tests in mathematics to separate students out by ability. This leads to the view that differentiated goals and aims are appropriate across the range of mathematical “abilities”, as manifested in mathematical achievement levels.

The Progressive educators aim for students to learn to be creative, to express themselves and to
gain confidence through learning mathematics. The aim to encourage the development and flowering of the whole person is ethically commendable. But overemphasised it is unrealistic because learning mathematics is to a large extent reproductive, mastering the knowledge of past generations through the practice and reinforcement of skills, as well as developing some competence in problem solving. Creativity is possible in school mathematics but it is a small component compared to the required mastery of knowledge and skills. In addition, all mathematics teachers must address school examinations and assessments, as these are major passports to enhanced life chances. Thus progressivism suffers from being individual-centred at the cost of not being socially aware and responsive. This is putting individual goods ahead of social goods, and doing so unrealistically. It is also difficult to implement in practice and there is little research evidence that progressive teaching programmes result in higher achievement or more positive attitudes in mathematics.

For Public educators, the main goal is the empowerment of learners as critical and mathematically literate citizens in society. Again these are worthwhile aims, which are good both for individuals and for society, since the promotion of democracy and social justice are ethical goods. However, there is a danger that the needs of individuals become secondary to social goals, and for education to become too politicised. The politicisation of education creates social conflict and opens the door to subsequent swings in the political orientation towards ideological or reactionary doctrines. In addition to the public educator goals, students need to develop their own individual interests and talents, as well as preparing for examinations, for the reasons discussed above. In developed countries there is little or no evidence of the success of Public educator programmes in school mathematics, especially since none of this type have been tested on a large scale.

However, it must be acknowledged that only the Public educators offer a set of aims for school mathematics with an explicit ethical dimension. Using mathematics as a vehicle for raising ethical issues in the classroom, including social justice for humans, care for animal welfare and care for the earth and the environment can only be good thing. Using real-world examples from such areas as a source of problems and modelling applications not only helps to develop student skills, concepts and strategies, but also motivates problem solving. Including ethical issues in the mathematics curriculum in this way provides the mathematics teacher with an additional asset. Thus the benefits go beyond merely adding ethics to the curriculum, they both enliven study and help to develop students as balanced and rounded human beings.

This justification raises the question of whether an ethical mathematics teacher should or must include ethical issues within the content of the mathematics curriculum. My opinion is that if this is done well it is a good thing, an asset to students and society. But to compel all mathematics teachers to include such content is problematic. For if politicising the mathematics curriculum runs contrary to the philosophy or beliefs of the teacher then until ethical content is mandated by law compulsion would not seem to be right. Furthermore, an unwilling teacher may not make the best case for ethics in mathematics and its applications. However, times may change. For example, in Australia a number of Universities including La Trobe have made sustainability education and global citizenship, which share some common ground with the Public educators' aims, a necessary component for undergraduate students in all subjects.

What this evaluation of the aims of these five groups nevertheless shows is that even though some of the aims are more ethically defensible than others, no single one of them can claim to be ethically the best and wholly good for all. Historically, the five groups proposing these aims have been in conflict, so each group has fought to increase the emphasis on their own particular aims in the overall outcome. Thus, since no one of these aims is the best on its own, a balance between them, a compromise, is desirable, in which the weakness of some are balanced by the strengths of others.

Over time it was not the optimal ethical compromise between the group aims that was adopted, but the relative power and dominance of the groups that determined the outcome. Of course the outcome has not been static through the years. At the beginning the Progressive educators and their aims played a significant role in the development of the National Curriculum in mathematics, since this was the dominant ideology of the mathematics educationists involved in its formulation. They succeeded in including progressive activities including investigational work, extended projects and problem solving in the mathematical National Curriculum and its assessment. However, over the course of the 1990s the influence and the inclusion of Progressive educator aims has been all but eradicated from the National Curriculum. Against this declining influence, in the late 1990s the National Numeracy Strategy emerged, which included more emphasis on mental mathematics and individual student reasoning which
supports the *Progressive educator* aims. But the net overall effect is that the emphases on progressive elements such as problem-solving strategies and investigational work have only survived insofar as they could be represented as applications of mathematics in such curriculum elements as *Using and applying mathematics*, thus more directly serving the aims of the *Technological pragmatists*.

The aims of the *Public educator* group were never reflected in the mathematics curriculum at any stage, and the aims of the first three groups have come to dominate. These are basic instrumental numeracy for the lowest attainers, practical mathematics and teaching to the tests for the majority and higher mathematics for the highest attainers destined for university or scientific professions. These are not optimal ethical outcomes. More emphasis on *Progressive educator* aims would better round out the personal development of students, enhancing their flourishing. More emphasis on the *Public educator* aims would empower students as critical citizens better able to contribute to and sustain a democratic open society and concerned with social justice and environmental problems. This is evidently an ethical good, not currently addressed in school mathematics.

**Paul Ernest is emeritus professor of mathematics education at The University of Exeter.**

**References**


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**In-sight**

Shane Johnstone, MT's artist-in-residence, offers a tribute to Trevor Fletcher.
Mathematics borrows from everyday language

Jim Thorpe explores everyday and mathematical language and hints at classroom consequences.

Mathematicians, claimed Goethe, are like Frenchmen: whatever you say to them, they translate it into their own language, and forthwith it means something entirely different.

Among my endeavours to support students studying Open University modules ME627 Developing geometric thinking and MU123 Discovering mathematics, I found myself reflecting on how everyday words also used in mathematics may obstruct, or, contribute to learning and I contemplated potential confusion emanating from dual or overlapping meanings. Not only words, the mixture of informal and technical talk may also be a stop or step to learning.

My interaction with students often starts with their email query. Lacking the quick-fire repartee of the classroom, my replies must at least implicitly invite dialogue. Students may one-sidedly terminate an email exchange; no captive audience, they may ignore me when they choose.

Miss-takes (miss-gives?) I think, vary in their capacity to confuse, and accordingly the ease with which the opaque may be clarified. Aware that this diagram (figure 1):

![Figure 1: Viewed by learner - named by teacher.](image1)

shows a right angle, a learner who describes this (figure 2):

![Figure 2: What might learners call this angle?](image2)

as a “left angle” displays descriptive power but mathematical innocence of how it is that mathematical terms are not so much descriptive as tied to mathematical properties. Unless unwilling to let go of their customary descriptive power, learners may readily accept our attribution of “right angle” to both diagrams as “left angle” is neither everyday nor mathematical usage. We might remark that “right” is short for “upright” in our attempt to drag learners away from their idiomatic description, saying what they see. By acceding to our dismissal of left angle, our learners have implicitly, or explicitly if told, begun to buy into the mathematical idea that properties of shapes are unaffected by position, invariant under isometries as the saying goes.

What presently concerns me is the clash in the minds of learners between the everyday and the mathematical, a conflict which may confuse, or lead to angry dismissal of mathematics as nonsensical. Further, I would like to know how to enable the new to be added to learners’ natural descriptive powers, but not in any way disparage natural competence, perhaps pompously with, “We use words more precisely in mathematics.” Should we allow learners to stay with left angle until encountering transformations and congruence?

We present to learners idioms commonplace among mathematics aficionados, but is our dialect any more transparent than colloquial turns of phrase are to foreigners learning our native language? About to return home at the end of WWII, Polish airmen lined up for a valedictory address by their leader to their RAF counterparts, their hosts for the duration. The Polish leader’s address ended with “… and may God pickle you all!”. A similar error (misconception?), innocent of context, is attested to by a translation into Russian of a journal article which contained the maxim, “Out of sight, out of mind”. At a later date the article was needed in English; the maxim was translated back as, “Blind and mad”.

We teachers seem to get away with the mathematical use of “property”, perhaps because it is semantically so distant from a home, or from one’s residential cash-cow; or is it because we just use rather than define “property”, referring to the angle-sum property.
of triangles, or the defining property of a tangent as a line touching a curve and its consequential property of being normal to a radius of a circle at the point of touch? … normal in the sense of perpendicular, not normal as in probability distribution, nor in topology.

When learners misinterpret mathematical terms, relying on everyday interpretation of what they perceive, their answer to “How many diagonals has this rectangle?” (figure 3).

![Figure 3: Rectangle displayed to learner.](image)

may well be “None.”, but when asked about this rectangle (figure 4):

![Figure 4: Rectangle tilted.](image)
a frequent answer is “Four.” David Pimm points out that, in everyday use, “diagonal” describes something sloping, in contrast to vertical or horizontal: But in mathematics “diagonal” is a property of shapes (Pimm, 1978). Rectangles have diagonals, even when none are shown, as in figure 3.

Mathematics adopts and adapts everyday words, so “symmetrical” becomes “symmetric” in statistics, though both words are used variously in algebra and in geometry. Learners may be nonplussed on encountering familiar words used as mathematical objects or concepts. In everyday speech we say, “A third of a bar of chocolate” to describe how much chocolate. In mathematics we have the concept of one-third, now no longer a description of quantity. Undergraduates, as Alcock and Simpson (2009) point out, may also experience uncertainty about mathematical use of everyday words such as “limit”, as well as confusion arising from their dual use in mathematics as process and as value.

Is $2^2$ a number, or is it an instruction to calculate one?

A word or phrase may point towards the mathematics or it may distract us. I asked a student to place a band to show the triangle in figure 5, describing it during a phone tutorial, and asked her:

“How may you find its area?”

![Figure 5: Band on a Geoboard.](image)

“Halfbasetimesheight”, she swiftly replied, to which I responded, “Could we relate the triangle to its bounding rectangle?”, which we may have both thought of as the diagram in figure 6, but I chose not to check. It was enough that the conversation continued to work. At least I think I know to what she was attending, but not in what way. Perhaps she saw the rectangle as two of the triangles put together, or the diagonal as the line creating the two triangles, or focused on an area relationship, and so on. “Yes”, replied the student, “the triangle is half the rectangle.”

The original “half”, which with sophistication we may describe as the multiplicative operator “one half of” was for her part of the ritual incantation of the formula. With the second “half” my student related two shapes, as geoboard work facilitates. In geometry, diagrams may be worked with, not merely provide springboards for algorithmic activity.

The semantic distance between everyday and mathematical meaning affects, I suggest, how easily learners cope with dual use. As illustration I will contrast the question, “Does a table have legs, or does it have columns?” with the notion of invariance. A table to eat off has legs, but a table in mathematics has columns. Confusing the two is unlikely because their application, albeit not their etymology, is so different. “Invariance”, on the other hand, is close enough in its everyday use and in mathematics to provoke confusion.
In everyday English, “invariant” simply means unchanging. In mathematics it is a bit different. It is about what is unchanging under certain conditions. An invariant, ubiquitous in mathematics, is a property that stays the same when some change takes place, for example, a geometrical transformation.

I was alerted to confusion with its non-mathematical use when a student erroneously declared that everything about the geometrical shape and the geoboard being used for the student’s task was invariant. Also, the student employed “variance” as the opposite to “invariance”. Sensible enough in everyday parlance but in mathematics “variance” is a term from statistics not the opposite of invariance.

Mathematical terms relate to their everyday cousins in diverse ways. In textbooks we see these definitions (figure 7):

![Diagram](image)

Figure 7: Interior and exterior angles of polygons.

although the following interpretation of exterior angle is plausible in the light of the provided definition of interior angle (figure 8):

![Diagram](image)

Figure 8: What might learners call this angle?

I see one text proclaims, “This is not an exterior angle”. (Heirs to Magritte, do texts in France announce, “Ceci n’est pas un angle extérieur?”)

Other terms may place even greater demands on learners’ capacity to comprehend the bizarre: “expanding brackets?” Suspension of disbelief?

In response to a task asking for the distance a ball is thrown and which is found from a quadratic equation with solutions –300 and 67, students who are aware that the negative value has to be ignored, sometimes write, “The distance is 67m. The –300 is negligible.”, unaware of, or ignoring the mathematical meaning of negligible.

Along with words with multiple meanings, the status of rules such as, “two minuses make a plus”, is problematic. Are they mathematical rules, or simply aides memoires?

A student addicted to rules asked me why -20 + -10 made -30 not the +30 she expected, a conflict with her rule about two negatives, remembered from earlier times. I suspect many students rely on prior learning, ignoring the course texts. I emailed my student this diagram (figure 9):

![Diagram](image)

Figure 9: Sketch sent to student.

“I understand from your diagram”, the student replied, “but that means my rule doesn’t always work.” Perhaps I did the student no favour, undermining her regular certainty.

Learning may at times be enabled by a mathematical term or a mnemonic; in contrast, exemplifying their potential to disable, learners may know “tangent” in trigonometry, but “SOHCAHTOA” obscures its link to tangent to a unit circle. I wonder if SOHCAHTOA is best avoided?

As well as learners’ discomfort with unusual use of their familiar words, we teachers may find that ambiguity obscures our on-the-spot assessment of understanding. Lacking our own well-worn identification of representation and data-spread my student’s designation of his boxplot as “symmetrical” may refer merely to its visible symmetry, not to its statistical meaning. A different student, however, left me in no doubt as to what she meant, although describing her own boxplot as “right-skewered”.

Both everyday and mathematical utterance figure in the classroom, so learners must cope with both. More than merely cope, we actually esteem non-specialist talk when we encourage our learners to argue points
of view, agree, disagree, without pressure to meet canons of formal reasoning. We may not, according to situation, leave it there in the shared agreement of friends, but take different or narrower paths from conjecture to proof, echoed in the mantra, “convince yourself, convince a friend, convince a sceptic.”

Adrian Pinel (see MT217) alludes to his way of shifting learners’ everyday descriptives to mathematical terms:

… whilst adding to their vocabulary subtly. They would say “turn” and I would say, “yes, turn, … rotate”. The next time they would say, “turn, rotate” and later just “rotate”. This approach led them to adopting my language, but with understanding … .

I wonder how Adrian reacted to his learners when at times they partially reverted to their former colloquial utterance? Not at all I suspect.

We are told that high attainers in mathematics cope better with the treachery of multivalent words. Is this because they are better able to discount their everyday discourse, their socially held knowledge?

If terms were used in only one way, learners’ lives might be easier but as Poincaré points out:

Mathematics is the art of giving the same name to different things.

In his review of *Infinity and the mind*, Dick Tahta reminds us that:

People say things multiply when there is increase. Mathematicians also say they multiply when there is a decrease (times half) or when neither increase nor decrease is in question (times a matrix).

I do not see that learners simply need to cope with mathematical terms, dismissing their everyday use. As it happens, mathematical texts employ both the technical and the informal. Learners may be told that a solution to a quadratic equation should not be written

\[ x = 3 \text{ and } x = 2, \]

on logical grounds, but as

\[ x = 3 \text{ or } x = 2, \]

as seen in the texts, although on the next line the text announces, “So the solutions are \( x = 3 \) and \( x = 2 \), returning from logical to everyday.

My examples, amongst other experience, lead me to believe that relationships between language and mathematics are not straightforward, a belief echoed by the observation:

Mathematical concepts and topics are not simply ‘ideas’, nor are they simply ‘definitions’ or techniques. Rather, the words are labels for a complex tapestry of interwoven thoughts and images, connections and links, behavioural practices and habits, emotions and excitements.’ (Johnston-Wilder S and Mason J, 2005)

I do not believe that direct instruction is in general an optimal way to address everyday and mathematical discourse in the classroom. Also, no approach should denigrate learners’ socio-linguistic heritage. The subtlety of prompting seems more respectful of the natural powers of learners.

Although I am relaxed about mixing ordinary with specialised language others appear less liberal. Thornton (1970) says, “any must go” in an article of that name, the dictionary definition of “any” allowing it to do duty for the mathematical “for all” as well as for “for at least one”. Thornton, however, is more subtle than this example suggests. He writes:

As with so many important aspects of the body of mathematics, the needs of the learner are in conflict with the conscience and convenience of the teacher. Which of the subtleties and refinements can be deferred until later, without actually misleading, in order to make the broad ideas accessible to beginners? (p. 221)

Perhaps Thornton has it about right, there is no single answer, decisions about everyday and mathematical classroom talk being, to turn a phrase, “horses for cases”.

**References**


Reflections on working with The Gang: A journey towards computational fluency?

Dave Benson explores ways to develop computational fluency in learners who lack confidence in their mathematical abilities.

Motivations and background

For many years, a question has troubled me: to what extent can students who feel significantly challenged by working with number build genuine and sustained confidence in dealing effectively with calculations that come their way? By "significantly challenged", I refer to the kind of student whose ability to deal with a problem or calculation in a secure way is, for example, undermined by uncertain knowledge of and weak fluency with number bonds, place value, multiplication tables or equivalences between fractions, decimals and percentages. A supplementary question I would pose would be whether, for some learners, a sense of belief in the prospect of progress fades.

In a persuasive article, Russell (2000) advances the view that “computational fluency” comprises three key elements: efficiency, accuracy and flexibility. She goes on to explore the importance of “connectionist” thinking in securing confidence in carrying out calculations. She urges us, as teachers of mathematics, to see procedures as "a web of connected ideas" and emphasises the importance of developing “mathematical memory” over memorising. Askew in MT174 and Thompson (1995) add to the connectionist picture when they explore the potential of learners to derive new facts from known facts. I became personally motivated to explore the scope and limitations of connectionist thinking for these students as well as their ability to “notice” and “see” the connections (see Watson and Mason, 2006).

Strategic approach

My approach was rooted in informal action research at one of our partnership secondary schools, with a group of four 12- and 13-year-old students, whom I and the head of department grew to refer to, fondly, as The Gang. From a cohort of around 190 students, we identified these participants who had persistently exhibited insecure numeracy skills and were positioned close to the bottom of rankings based on departmental assessments.

My initial conversations with The Gang provided me with the opportunity to share the principles of my approach with them as well as the dispositions I was looking for in them. I made the following points clear:

- They needed to try and believe they were capable of becoming more confident and competent in their use, understanding and recall of number.
- Consistent engagement and honest reflection had to be given. They would be expected to keep their school planners and the record sheet (see figure 1) up-to-date.

![Figure 1: A record sheet.](image)

- It was important to persevere when the going got tough and to try not to give up.
- They needed to commit themselves to independent and consistent practice at home.
- They were expected to look after their resource packs carefully and bring them to every session. Packs consisted of multiplication table practice cards, double-sided complements to 100 cards (for example, 62 backed with 38), a weblink card (see figure 2), a connection cloud mini-whiteboard and pen, a set of fraction-decimal-percentage matching cards and a Gattegno (place value) grid with transparent counter (see figure 3).

![Figure 2: A weblink card.](image)
Reflections on working with The Gang: A journey towards computational fluency?

• Although the project was a joint effort between me and them, they needed to accept responsibility for their learning. They would be expected to try and develop proactive independence and autonomy.

• They needed to see themselves as “mathematical detectives” who were looking for connections between numbers and operations.

The enquiry was structured into one-hour sessions shown in Table 1 below.

Reflective narratives on two of The Gang

Student A

The following points emerged from my initial interview with student A:

• She felt confident with column arithmetic for addition, subtraction and multiplication, but still needed to use fingers to support.

• She disclosed she could rarely deal with any calculation mentally, no matter what the operation.

• Division posed particular problems for her in all respects, “unless the numbers fit within the times tables I know.”

• “I struggle with fractions and decimals and do not get percentages at all.”

My initial assessment of her ability to bond integers confirmed her self-analysis. For example, she was unable to bond 6 or 2 to 10 without the use of fingers. Unsurprisingly, complements to 100 generated even more uncertainty. Even with the support of a 100-square, these proved challenging. For instance, counting on in 10s proved unreliable and the response of 39 as a complement to 100 for 71 was typical. Hesitancy and uncertainty pervaded all responses.

Over the course of the project and despite these gaps in her knowledge and understanding, Student A offered willing, if at times inaccurate, verbal contributions to our discussions. She also displayed a genuine desire to make progress. Some clear evidence emerged that she had succeeded. For example, her work (see figure 4) testified her ability to use the potential of connectionist thinking to solve problems. Neither 60 x 6 nor 60 x 60 posed her a problem. Verbally, she also confirmed that she knew 3 x 6 = 18. From that, she was able to deal with a chain of problems: 30 x 6, 31 x 6 and 310 x 6.

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Table 1.
student. Figure 5 shows how, particularly in the first section of questions, she resorts perhaps too quickly to a column method, making errors, with little appreciation of place value. That said, the second section provides some, more hopeful, evidence of progress with her response to the question 0.17 + 67. Here, observation confirmed she generated the correct answer mentally before checking her answer using an accurate column method.

Although evidence from an early session suggested he was limited in his ability to see the relationship between calculations in a consistent way, the connection cloud in figure 6 for 6 x 6 = 36, suggests that he was able to make clear progress.

In his initial interview, student B assessed himself as being confident with mental addition, subtraction and multiplication of integers but under-confident with division, fraction, decimals and percentages. His initial assessment confirmed his instant recall of complements to 10. Significantly, however, when challenged to find complements to 100, he struggled when the given number was not a multiple of 10. Further examples confirmed he was dealing with the tens-digit first and was harbouring the misconception that the tens-digits needed to add to ten. Once I had shared my observation with him, however, he began to offer correct responses. The outcomes raised questions about the accuracy of his self-assessments offered at this stage.

Over time, student B displayed an ever-increasing ability to see the relationship between questions and developed insight into how this may be advantageous in solving more complex problems. Although less secure when decimals became involved, figures 7 and 8 highlight how, even when faced with more significant place-value issues, he began to apply his skills and knowledge more confidently. Although this obviously generated potential scope for error, he maintained an interesting determination to deal mentally with the challenges. His response to 1.7 + 0.67 illustrates this was not always plain sailing for him.
Implications and next steps

Perhaps, my most significant observation would be the importance of creating positive dispositions to learning and the belief, both on the part of the student and the teacher, that progress can be made. A readiness to persist when the going got tough also proved to be key in securing progress with in-school activities. Resilience did not appear to be an issue for these learners and responses from all four to school-based activities consistently suggested that they were prepared to be tenacious even when faced with the challenges of dealing with calculations involving decimals.

However, my experience testified that progress was consistently rooted in a readiness not just to engage positively in school but also to find the motivation to extend their efforts beyond the classroom. The student who claimed to have worked most regularly and consistently at home, student B, made most progress. In his final assessment interview he managed, independently, to complete 22 complements to 100 when, in his initial interview, he had struggled to succeed with any until I supported. He also demonstrated an ability to bond numbers with one decimal place to 10 in a way that, at earlier stages in the project, he simply could not. Conversely, the student, who probably worked least reliably between sessions made least progress. Clearly, responsibilities for generating positive attitudes to learning rest with both the learner and the teacher. However, the project did not consider the potentially key role of parental support in any focussed way.

All four students confirmed that they had found the pack of resources supportive. They commented on the engaging nature of the IT resources and on how these had encouraged them in particular to practise multiplication tables as well as to think more confidently about decimals. The connection cloud board also became a firm favourite. Arguably, this proved to be the key resource for scaffolding learning. Consistent evidence emerged that this resource enabled students not just to “see” (Watson and Mason, 2006) the connections between numbers and calculations but also, crucially, to take control of creating connections in order to solve problems. Final assessments confirmed that all four had become much more aware of the potential of using known facts to generate further facts using the connection cloud boards to support. From my perspective, the Gattegno (place value) grids contributed significantly to the students’ appreciation of the impact of multiplication or division of decimals by 10 or 100. In a simple way, it enabled me to focus the students’ attention on the movement of digits rather than on the appearance or disappearance of zeros.

On reflection, my experiences of working with The Gang allow me to offer a hopeful response to my original questions about students who feel “significantly challenged” in their relationship with number. Within one narrow focus of Russell’s interpretation of computational fluency, these students could unquestionably secure a stronger fundamental knowledge base and, crucially, build higher levels of self-confidence. That said, my question also included the word “sustained”. That clearly goes beyond the remit of this project and remains to be seen. However, from the perspective of a secondary school, my observations appear to argue a strong case for a targeted and systematic approach over time if progress is to be sustained.

A final thought from one of The Gang. On being challenged to calculate 41 x 7 from 40 x 7 = 280, a fact she had generated from being given 4 x 7 = 28, she proceeded to use short multiplication to calculate the correct answer of 287. On completion of the written calculation, she looked at me, smiled and commented, “I have just noticed. I could have just added 7.” Indeed, there is hope.

Dave Benson is Mathematics Education Coordinator at the University of Derby.

References


Drug calculations: A non-formulaic approach

Lois Rollings reports on her experience supporting nurses with calculations involving the amounts of drugs to administer.

I have been reading Mathematics Teaching since I joined ATM as a PGCE student 35 years ago. It is always a particular pleasure to find articles in it that resonate with my own thinking. In my current job I spend a lot of time helping nurses with drug calculations. A typical problem is as follows:

Your patient is prescribed 75mg of a drug. The stock elixir is labelled 20mg/2ml. What volume do you administer to the patient?

Before reading any further I invite you to solve this problem (without a calculator) and to note how you did it.

(Editors note: We wondered why drugs are labelled as 20mg/2ml rather than the more calculation friendly, 10mg/ml. Lois conjectures that this is because this particular drug comes in 2ml vials. Similarly drugs may be labelled as “per 5 ml if they are to be taken using a 5 ml spoon.”) Reading MT262, I realised that the kind of percentage problems described by Chris Carter have parallels with such drug calculations as they are both about ratio and proportion. Also, my students, like his, often want to take an algorithmic approach whereas I want them to be more discerning.

A particular problem is that what the nurses in Hoyles et al. (2001) study describe as “the mantra” is pervasive in the profession and in textbooks and journal articles. The words in this mantra or “nursing formula” vary. They might be, “What you want over what you have times what it’s in” or, “Need divided by have times stock volume”, with the handy NHS mnemonic. Interestingly what Hoyles et al. found was that few nurses used this mantra in practice. However, as Wright (2013) points out, this paper was “ironically” published in an education journal, and so may not have been read by those in the nursing profession. Although there are some exceptions, many articles in nursing journals take the formula as the starting point for calculations, as do most of the text books I have seen.

There seems to be a view that encouraging the use of the formula will reduce errors in drug calculations and I have even come across a suggestion that a standard verbalisation should be used. I would argue that relying on the formula means that students are not flexible in their thinking, which can lead to an instrumental understanding. It can also sometimes make the arithmetic more difficult.

As described by Hoyles et al. (citing Vergnaud 1983), there are five ways to solve a problem such as my example above:

- noting that the stock concentration is equivalent to 10mg/ml and so to get 75mg we need $75 \div 10 = 7.5ml$. This is described as a functional or between-measures approach as we need to think in terms of mg/ml.
- using a scalar approach or staying within measures. This can take three different forms:
  - decomposition might involve doubling and halving in order to build up the required amount. So there will be 40mg in 4 ml, 10 mg in 1 ml and 5mg in 0.5ml, so we need $(4 + 3 + 0.5) ml$.
  - the scalar unitary method would involve noticing that to get to 75mg from 20mg we need to divide by 20 and multiply by 75 and so we need to perform the same operations on 2ml to get the required volume.
  - the scalar operator method would use the fact that $20 \times 3.75 = 75$ so we need $2 \times 3.75 ml$.
- using the rule of three. If the required volume is $x ml$ then $20x = 75 \times 2$.

I have noticed that the rule of three is often used by students from mainland Europe but rarely by those schooled in the UK. The European students often refer to it as the “ratio method”. I am not sure if they understand why their method works but they seem to be able to spot quickly that the situation demands it. The rule of three is, of course, essentially equivalent to using the formula. Do you recognise which of these you used, or could you just ‘see’ the answer without thinking about it consciously?

I am aware that which of the methods outlined above
Drug calculations: A non-formulaic approach

will be easiest will depend on the numbers in the problem. As mathematics specialists, we will spot the “nice” numbers and adapt our approach accordingly. It is my experience that students are not always able to see which numbers are “nice”. Indeed, we might be happy to use the formula as we can then easily interpret the result in terms of the context, but this can be another difficulty for students.

Wright (2013) claims that the formula is more likely to be used by less confident students. I find this concerning, as I feel that they might be making things harder for themselves. Using the formula requires students to be confident in manipulating fractions and undertaking division without a calculator. They need to appreciate how and when to cancel and to understand the order of operations.

I have worked with students who are happy to “cross off the zeros” and halve the numerator and denominator but not to cancel by other factors or to reduce any numbers that are not directly above each other. Several authors cite errors involving doing the division “the wrong way round”. In a short research project as part of a master’s course, I found that the adult learners I investigated could not always link the different formulations of a problem such as 6 ÷ 3, how many 3s in 6?, 6/3 and the set-up for the bus-stop algorithm. There are also other formulae for different situations and I have known students try to memorise all of them without spotting their similarities or understanding how they work.

As Barbara Ball remarks, in her conversation with Laurinda Brown in MT262, articles which describe perfect lessons are not helpful. I have no easy answers to how to increase nursing students’ confidence and accuracy in performing drug calculations. They are understandably anxious. If they do not pass their written university drug calculation test then they cannot qualify and, of course, errors in practice can have serious consequences for their patients.

I cannot pretend that the nursing mantra does not exist. Students will only have to open a text book or speak to a colleague when on placement to become aware of it. However, I will continue to encourage them to think beyond it and one thing I might try is to adapt the ‘I’ method described by Carter in MT262. It should help them to get a sense of the size of the answer they are looking for.

In the meantime, I look forward to receiving the next issue of MT.

Lois Rollings supports students at Middlesex University.

References

Trevor died last year at the age of 96. His passing seems like the passing of an era. He was not quite a founder member of ATM, joining a couple of years after the start.

Film making and the connection with teaching aids, part of the Association for Teaching Aids in Mathematics (ATAM), was what attracted him. He was quickly active on general committee, just about everyone had to be in those heady days. When Mathematics Teaching started, he became the first editor.

In 1958, Caleb Gategno, with his extensive international network, secured an invitation for Trevor to go and work with Norman McLaren and René Jodoin at the National Film Board of Canada. So, that autumn, Trevor, his wife, Beryl, and a very young Robert set off by boat for Montreal. In Canada he worked with René Jodoin on *Dance squared* [https://www.youtube.com/watch?v=yXL4DP_3dJI](https://www.youtube.com/watch?v=yXL4DP_3dJI).

One of Trevor’s interests at the time was how to make group theory more visual. It is worth watching *Dance squared* with that bit of extra knowledge. In an interview ten years ago, Trevor claims that he was very bad at visualisation and made films to help him understand what was going on in the algebra ([http://www.icmihistory.unito.it/interviews/fletcher/fletcherpart5.m4v](http://www.icmihistory.unito.it/interviews/fletcher/fletcherpart5.m4v)).

While in Canada he also made *Four line conics*. It extends the ideas he had been working on developing the ideas and techniques of Jean Louis Nicolet in his three ATAM films *The cardioid*, *Four point conics* and *The Simson line*. All three are available on the ATM website. ([https://www.atm.org.uk/Trevor-Fletcher-Films](https://www.atm.org.uk/Trevor-Fletcher-Films)).

In 1963, at the height of the ‘New Maths’ era, ATM secured a grant to support a group of members to spend a week in a residential centre writing secondary maths teaching materials for new maths topics such as transformation geometry, functions and group theory. This enterprise broke new ground. Before the week, Trevor produced a list of possible chapter headings and each of the eighteen people took on the task of writing material for a particular chapter during the day. In the evening the writing was shared around and a critical review with no holds barred ensued.

After the week, Trevor, as editor, took the last drafts away and worked them up into *Some lessons in mathematics* (1964). At this stage, chapters went back and forth to a number of sub-editors, one of whom was David Wheeler. Much later, Trevor mused that when it went to David he added in a whole lot of mathematics education theory removing some good mathematics. When it came back to Trevor, he cut out much of David’s work to get the mathematics back in. This reflection is an insight into the tensions that were around in the heady days of the sixties and the start of the reforms in mathematics education.

For those of us who joined ATM later, he is remembered as the Staff Inspector for Mathematics (HMI) for his contributions and tireless quiet work behind the scenes in setting up and supporting the structure of the Cockcroft Committee. He ensured that there was representation from both ATM and MA and was then quietly around in an ex-officio capacity at just about every meeting.

Before Cockcroft, many mathematics classes started with the teacher solving a problem on the board while the pupils watched and then the pupils opened their textbook and worked through a long list of similar problems. Trevor wanted to break this cycle. One of his techniques was to say, “A problem is no good unless you can spend a whole lesson discussing it”. He collected good problems and many people have recollections of early Trevor sessions when he poured out one problem after another while we frantically tried to make notes.

Trevor’s fingerprints are all over Cockcroft. For many years the most quoted paragraph was 243. How on earth do you organise a book of many hundred numbered paragraphs so that the principal message you want to convey is associated with 3rd? The mathematician inside many teachers would notice that number.

All through the time that Trevor was Staff Inspector for Mathematics he continued his interest and involvement in mathematics on the international stage. He was interested in, for example, German and Russian research in teaching and learning. He was not however ever going to say the answer to our problems lay in looking at what happens in Russia or Germany and importing it.
Geoff Faux writes:

I was appointed as maths adviser in Cumberland in 1972. Trevor lived just across the country in Darlington. Computing in mathematics was in its infancy. Calculators were prohibitively expensive. Trevor put me in touch with Bob Lewis and the three of us put on a residential course for secondary teachers on using computers in mathematics. I had two programmable machines in county. Bob arrived with a vast piece of equipment in the back of his car. Trevor arrived with a series of problems. We never managed to make telephone contact with the mainframe down in London but we tried. At that time there were still three primary schools in Cumberland without a telephone. Urgent messages from the office went by telegram and we were trying to introduce computers into secondary schools. However, a consequence was that Trevor’s problems ricocheted round various Cumberland maths departments for the next couple of years.

Another happy memory is bound up with Trevor coming over to give sixth form maths lectures. We arranged these each year for a summer term afternoon and then the two of us would sneak of for three or more hours on a local fell. Walking the high fells of Lakeland, along with classical music and playing the piano were Trevor’s real passions.

David Cain writes:

I was planning an Easter ATM conference in Northampton around the concept of Minimalism. As part of the conference planning, Geoff Faux and I went to Darlington to see Trevor. One reason was that he was writing rich programs for Logo and I wanted him to do something at the conference. After a pause for thought he turned down my request, but he hoped we had a successful conference.

More importantly for me, he is the reason I joined ATM. I was a teacher in Silloth and I was told we had an HMI coming to inspect. I was the music teacher, but this year I had been asked to teach some mathematics. He came to one of my lessons. At the end he said, “That was very interesting. Why did you ask that boy to explain why the area of a circle increases four times if the radius is doubled? Why not tell him?” I said that I know that he knows but he must think about it. He then said, “Thank you very much” and left. Later I found out that he suggested to the Mathematics Adviser that I should join ATM.

Leo Rogers writes:

I was an undergraduate at Sir John Cass College (1959-1961) and found myself in Trevor’s class. I remember particularly the way he taught the analytical geometry course. After my A-level course, it was a revelation. Problems were posed and possibilities discussed among the class and then we were sent away to sort out some solutions. Our solutions and failures were reviewed and then Trevor would reveal other solutions, opening up new and fascinating possibilities and extensions. During the lunch break he would often show us films. I remember some of the Nicolet films, but also The cardioid, and Four point conics which at first were quite baffling. I know we saw them more than once and here we could see the enormous possibilities of the use of dynamic images in teaching. On reflection, it seems to me that much of his approach had been inspired by the Nicolet films.

CUISENAIRE – FROM EARLY YEARS TO ADULT

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The 1960 meeting of the International Commission for the Study and Improvement of the Teaching of Mathematics (ICMI, now) took place at Krakow in Poland. The theme for study was mathematiques de base. It is not the function of the commission to reach formal conclusions and present agreed reports; it aims to raise questions in the minds of its members and to provoke thought on aspects of mathematics that they often do not have the opportunity to consider throughout the rest of the year. The members attend as individuals and not as delegates from their respective countries.

The phrase mathematiques de base can be understood in many ways; to some it means the logical foundations, to others the utilitarian topics which should be in the school course for every child, to others again it might mean the framework of elementary ideas, which is the key to the pursuit of the higher levels of the subject at its present state of development. It was the aim of the conference to understand and to reconcile these many aspects.

The subject was introduced by Professor Papy of the University of Brussels, Vice President of the Commission. He explained how recent progress in mathematics had made necessary a reappraisal of its logical foundations. Traditionally mathematics developed by adding to the Greek body of knowledge, often giving to the pupil concepts which were weaker, and less general, than those which he could have assimilated. But modern mathematics demands a relaying of the foundations, and the mathematiques de base of the twentieth century is radically different from that of earlier times. The main outlines of the subject seen from this modern viewpoint are analysis; algebra; topology; and set theory.

Professor Papy continued by showing how, in conformity with this scheme, elementary teaching of the subject could be based on the ideas of set theory, which provides a language for the subsequent development of many other branches of the subject. He showed how a particular kind of graph can be used to illustrate relations within a set, and how pupils can learn to appreciate logical structures and perform calculations with graphs of this kind. (Papy’s graphs are a rich concept, related to Venn diagrams in logic and to the diagrams which Cayley used in group theory. Later the conference saw his theories in action in the classroom.) We often confuse difficulties of verbal expression with difficulties of logical abstraction, and the use of graphical methods of reasoning enables these two difficulties to be separated.

After some sessions that were largely occupied by our Polish colleagues with a discussion of a new syllabus being proposed for their schools, the main business of the conference proceeded in two ways, through discussion groups and demonstration lessons. The main function of discussion groups is to help individuals to clarify their own thoughts, and there is little profit in detailed reports of their activities; but the demonstration lessons at Krakow raised many matters of general interest, and so most of the rest of this report is devoted to them.

It is easy to conceive a demonstration lesson as a lesson in which ‘someone who knows’ shows how it should be done. Many of the demonstrations which our own Association has arranged over the last few years have been of this kind, but a lesson can also be an experiment in front of an audience. A number of the lessons at Krakow were of this second type; not merely because of the incidental difficulties of language, but because the teacher was deliberately developing an unknown situation so that all who watched could learn from it, and learn from the mistakes and from the response of the class to unusual stimuli. This is a wider concept than a demonstration by an expert of something that he has fully worked out, and which he produces off pat; because the aim is to establish conditions from which new knowledge of the science of teaching mathematics can arise, and not merely to transmit certain pieces of mathematics by technical means, which have been decided upon previously. We should make a special effort to promote this type of demonstration as well. They are, of course, very much harder to conduct than the more conventional kind.

Professor Papy’s lesson was given to a class of
sixteen girls, with the aid of an interpreter. His material was quite new. The pupils were asked to represent themselves on the board, which they did by drawing a set of crosses, one for each of them. Then they were asked to add crosses for their brothers and sisters, and then to add arrows denoting the property “has a sister”. The problem was pitched at a level that they could understand, but thought was needed to solve it and to master the new notation which, in a great measure, they themselves devised. Some pupils’ mistakes showed that relations could sometimes be deduced even when the people were not known, because the relations concerned had properties of (partial) symmetry and transitivity. When this was understood, Papy drew a fresh diagram of an imaginary situation in which only some of the arrows were drawn, and the class had to decide what further arrows could be deduced.

This lesson was an obvious success. The topic provides a good exercise in logical reasoning, it gives an intuitive introduction to ideas which are fundamental in modern abstract algebra, it could be tackled, with obvious modifications, over a wide range of ages, and it is not in the least dependent on traditional arithmetic. In Belgium, Papy is working out a full course on these lines, working with Ecole Normale pupils, girls of fifteen and over, often with reputedly low mathematical ability (sic), who are training to be teachers. It is clear that they will be able to use much of the material which they are doing on theory of sets and topology with even younger pupils.

Dr. Gattegno gave a lesson in which he used the Nicolet film on the generation of conics as the loci of the centre of a circle passing through a fixed point and touching another circle. His questions were put with the help of an interpreter. As an experiment in communication it did not succeed. Dr. Gattegno has well known views on those who consider that the children are to blame if they do not learn when something is shown them; and this makes his lesson harder to appraise. The children were clearly on most unfamiliar ground. It was irrelevant that they ‘had not done the geometry’ leading up to the situation; the difficulty was that they just could not believe that they were wanted to describe the mathematics they had seen, in their own words, using their own judgement. They appeared to have been conditioned to an authoritarian type of teaching, knowing that a certain type of answer would please the teacher, and they were completely at a loss when they had to adopt their own standards of truth, stating what they had seen for themselves rather than repeating what they had been told. This lesson made no progress at all, in a forward direction, but some spectators may have seen that if further progress is to be made with many pupils who have followed the school course to the age of fifteen or sixteen the conditioning which they have undergone in the previous years must first be removed, as authoritarian teaching is inimical to true understanding. This lesson showed no one ‘how to do it’, but it was an experiment that led to a conclusion.

The first lesson by one of our Polish hosts was in great contrast. Mr. S. Turnau was working with pupils he knew, giving the next lesson in the course they were taking. There were no language problems, everything was as effortless and smoothly planned as one could wish, and it covered previously decided ground. (It was interesting that after the lesson, in private, Mr. Turnau expressed the fear that he had perhaps been wrong in giving a lesson that was a demonstration and not one that was an experiment). His class were given a geometrical pattern to study, photographs of a wallpaper design. It might have been a step nearer reality, and also more dramatic, to have given them an actual piece of wallpaper, but this is a detail. The questions were directed towards finding the elements of symmetry of the design. How did these elements combine with one another? Which formed subgroups? Which families of symmetries did not form groups? Can you draw another unit cell with the same symmetry? Ten years ago it was most difficult to find an account of these ideas in English, yet this demonstration showed beyond any doubt that sixth formers can do group theory in a constructive and most worth-while way. A large class of children from the earlier years of the secondary school assembled for a lesson from Madame Krygowska, who had made most of the local arrangements for the conference. Niadame Krygowska is the leader of the movement for the modernisation of mathematical teaching in Poland, she has an imposing presence, and this lesson with eight or nine blackboards being used by the children resembled an oratorio performance under a virtuoso conductor. The subject was convexity, and the lesson was first rate. Ten children drew figures of their own-geometrical figures, doodles, faces and other recognisable objects. They were told to shade the interior of their figures. Then they were asked, “Take two points and draw the line joining them. Is this in the interior of your figure or not?” Clearly, different cases arose, and so the idea of convexity,
and its precise definition, were born. The notion of the convex hull of a figure followed, constructed in many cases by placing a piece of elastic round plywood figures that had been stuck on to a baseboard. Once again we had a lesson that was clearly modern mathematics, which caused the class to think, and which would have found no place in the traditional syllabus.

I was late for the lesson given by Senor Galli of Uruguay, and for a while I did not realise that it had begun; for in a very few minutes he had managed to get the class thinking hard about a problem, and they were busy at it. He gave the lesson on the division of space into regions by different numbers of planes which is to be found in Vol. I of Polya's Mathematics and plausible reasoning. Knowing but a few words of Polish, Senor Galli sought to make contact with his class by speaking a mixture of all the languages that he did know. His interpreter seemed taken aback by the technique, but the class managed easily enough. I have never known this lesson to fail, and this occasion was no exception. The class worked hard on ground that was new and where even the type of reasoning was unfamiliar. The lesson was marred only by the teacher’s determination to finish in n dimensions or bust in the attempt! The final stages attempted too much too fast; but many teachers who watched will want to try the lesson for themselves.

Dr. Dienes's lesson was with young children of seven or eight. He used the teaching material of wooden blocks, which he has devised and tested out in England, and which is to be the subject of a further extensive programme of work in the near future. The children were given a pile of blocks and set the task of forming other piles containing (in different cases) four, five or six times as much wood as they had been given. They then had to try to make an equivalent pile using as few pieces of wood as possible, given a large stock in the middle of the table to choose from. All this was aimed at cultivating an appreciation of different number bases and an understanding of ‘carrying over’. Not being fixed to the base 10 the children were, effectively, learning to manipulate algebraic polynomials at the same time as they were mastering one of the early stages of arithmetic.

The lessons illustrated various aspects of the central theme of the Vice President's discourse. Many of the parts of mathematics which are considered modern and advanced can be understood by much younger pupils, they are basic in the part they play in the structure of mathematics as a logical discipline, and basic in the psychological development of the mathematician. These ideas must be better appreciated, and become part of everyday classroom practice.

Since this conference took place in Poland it inevitably raises the questions “What is it like behind the Iron Curtain?” and “How does their teaching differ from ours?” I have ignored these matters in the earlier part of this report, because it is necessary to put them into perspective. Throughout the conference teaching mathematics came first, and it was only a secondary matter that we came from four countries of the East and five or six countries of the West. Our Polish hosts took immense pains to make our visit enjoyable and to show us something of their country. The University at Krakow provided excellent facilities, and we received every consideration.

Polish teachers are faced with an enormous task of reconstruction; there is the memory of the occupation when all but the most elementary teaching was prohibited, and when teachers ran clandestine schools under conditions of unbelievable difficulty. The country has new frontiers, they have few economic advantages to begin with, and they have the challenge of neighbours with highly developed technologies on both sides. All these circumstances make their problems different from our own; but if a criticism may be made in the friendliest possible spirit, it did seem that in spite of a revolution and a social philosophy which takes immense pride in being progressive, the outlook of the teachers we met (with the most notable exception of Madame Krygowska and her pupils in Krakow) was very old fashioned. Furthermore, we found nothing new in psychological knowledge or theory which referred to the teaching of the subject. In mitigation of this criticism it should be remembered that the permanent members of the International Commission are severe judges, having come together over the years because of their reforming zeal and because they find insufficient sympathy for their point of view in their own countries.
Paul Killen reviews *Denominator* by Jonny Griffiths and *Millions, billions, zillions* by Brian W. Kernighan.

*Denominator* by Jonny Griffiths. *Denominator* is available from [www.denominator.co.uk](http://www.denominator.co.uk) for £10.00.

Some of us may describe ourselves as teachers of mathematics whereas others may prefer to think of ourselves as teachers of children. In this book, Johnny Griffiths, although clearly someone who loves his subject, clearly evidences that his whole career was based upon teaching children and his love of getting children to learn. The fact that the focus of this book is often the mathematics classroom, should not put off non-mathematics teachers from reading it. All teachers will recognise the many anecdotes that Johnny recalls and will have similar tales to tell. The book is about how children will always respond in a way you did not expect, or to put it simply, never underestimate the ability of a child to misinterpret what you say.

Within the book, Johnny integrates, I am sure he will like me using that word, a number of the articles he has written over the years for both *Mathematics Teaching* and the *Times Educational Supplement*. They are beautifully linked together with a real pace. *Denominator* is a reflection upon an extensive career set out in a non-linear pattern. We are led from the end of his teaching career to the beginning and back again in a random, non-chronological journey. Essentially, Johnny is trying to decide, “Was I any good?” This is every teacher’s dilemma. One moment we can think we are the most talented and gifted soul ever to have been employed to teach a group of children and in a moment be transported to the pit of despair, realising that the children have no idea what we are talking about and know less on leaving the lesson than they did at the start of it.

Our journey recalls Tower Hamlets in the 1980s. Was it really that bad? Or are all our memories only the extremes of what actually happened? I like the story of a teacher growing cress on the damp carpet to prove to their management that there was something wrong. The reminiscences of the *Secondary Mathematics Individualised Learning Experiment* (SMILLE) scheme actually made me smile. Johnny’s first year was, as it was for many of us, characterised by a persistent worry about his classroom discipline and why children’s behaviour suddenly improved so immediately when one of his colleagues appeared. Johnny survives this first year only because he has been told so many times by those in the know, that things will get better in the next year. They did not.

I love that Johnny is continually compromised in his attitudes to any child not totally engaged in their work. He alternates from a sense of, “Don’t you realise you are wasting this unique opportunity that will allow you to radically improved the quality of your future life?” to, “Well why should you want to learn mathematics, there is no point to it anyway?” I particularly enjoyed the time Johnny invites two of his old students back to talk to his current class. Both students are studying for a PhD in mathematics. Johnny cannot decide whether they only got to that level of education as a direct result of his amazing input during their early school years or whether it did not matter who taught them when they were younger, they are both so able, his input was immaterial.

The book is organised into short sections that make it easy to access and easy to read. However the book is not just a compendium of fun and comic anecdotes. It provoked me to think about a number of issues in relation to the purpose of our subject in the 21st century. One section considers the many advances we have seen over the past 40 years in terms of technology and resources, yet the way we formally assess our students’ understanding as well as the types of questions examinations ask, has just stood still.

Towards the end of the book Johnny hints at some of the medical issues he has had to address during his career. I do not know Johnny so this came as a surprise, but by that time I was so invested in what was being said, that I wanted to find out more. Perhaps Johnny is saving this for a future book? We hear today a great deal about anxiety in students and how schools should be responding to issues in children’s mental health. Yet very little is said about the same in teachers. We all know how stressful our job is and it is clear that different demands upon schools and teachers have added significantly to this stress. So an insight into this from the teacher’s side of the desk would seem a natural next step for Mr Griffiths. I shall wait with anticipation.

Brian Kernighan is a professor of computer science at Princeton University. In this book he provides the reader with the basic skills to interpret statistics that will enable us to be “intelligently skeptical (sic)” about the numbers that surround us.

May I just temporarily interrupt my review at this point as my first two sentences give an indication of, what for some, may be a somewhat less appealing aspect of the book. It is an American publication, written for an American audience with consequential American spellings. So in addition to “skeptical”, brace yourself for “recognize” and plenty of “liters” and “meters”. If this would be too much of a distraction for you, stop right here. If you can live with US spellings … read on.

The book is accessible to all. It is certainly not something written for mathematicians only. It will really appeal to anyone who enjoys the BBC Radio 4 programme More or less, as essentially it is doing a similar job. Similar works to this include Tim Harford’s The undercover economist or Innumeracy by John Allan Paulos.

There are perhaps two elements to the book. Firstly, the writer wants his readership to be able to make intelligent estimates of numbers around us. Here, he stresses that for everyday purposes approximations are good, and rough approximations are often good enough. Secondly, are we able to spot dubious statistics? Here we are provided with many real-life examples where the word million had been used instead of billion and vice versa. We are given some steer on how to think about these quantities, although I think this task is actually better executed in other works. Later, however, is a useful overview of how different prefixes are misunderstood as in megabyte, gigabyte, terabyte and so on.

One of the most interesting sections is how approximate numbers are often stated in our media to unnerving accuracy. This is referred to rather deliciously as “specious precision”. For example, Apple is reported in the Daily Mail as having recovered 2,204 pounds of gold from its recycled products. I also found the discussion on the size of a barrel of oil most enlightening. Other chapters in the book, which are perhaps more than familiar to teachers of mathematics, consider the dangers in misleading graphs, sampling bias and showing one-dimensional growth with two-dimensional diagrams.
Aims of ATM

The Association of Teachers of Mathematics aims to support the teaching and learning of mathematics by:

- encouraging increased understanding and enjoyment of mathematics.
- encouraging increased understanding of how people learn mathematics.
- encouraging the sharing and evaluation of teaching and learning strategies and practices.
- promoting the exploration of new ideas and possibilities.
- initiating and contributing to discussion of and developments in mathematics education at all levels.

Guiding principles

The ability to operate mathematically is an aspect of human functioning that is as universal as language itself. Attention needs constantly to be drawn to this fact. Any possibility of intimidating with mathematical expertise is to be avoided.

The power to learn rests with the learner. Teaching has a subordinate role. The teacher has a duty to seek out ways to engage the power of the learner.

It is important to examine critically approaches to teaching and to explore new possibilities, whether deriving from research, from technological developments or from the imaginative and insightful ideas of others.

Teaching and learning are cooperative activities.

Encouraging a questioning approach and giving due attention to the ideas of others are attitudes to be encouraged. Influence is best sought by building networks of contacts in professional circles.

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