Mathematics Teaching is the journal of the Association of Teachers of Mathematics (ATM). The ATM was formed in 1952 to encourage the teaching and learning of mathematics by relating closely to the needs of the learner. The aims and guiding principles of the ATM can be found on the back page of the journal.

Mathematics Teaching does not seek to conform to an official view on the teaching of mathematics, whatever that may be. We wish to encourage contributors to express their personal views on the teaching and learning of mathematics. We believe that everyone has a contribution to make, experiences and insights to share. Whether practical, political, philosophical or speculative, we are looking for articles that reflect on the practice of teaching mathematics. We aim to publish articles that will be of interest to the breadth of our membership, as well as a balance between those derived from research and those from practical experience. We see writing as a powerful tool for professional development and critical thinking and see Mathematics Teaching as a space for this development to take place.

The Editorial Board hope that:

- Mathematics Teaching is a space to share investigations into our teaching in some detail. Not generalising too much, but speaking for ourselves and of our experience. Never trying to speak for all.
- Mathematics Teaching honours the different ways that people know, do and engage with mathematics.
- readers will 'learn' from engaging with the articles, in trying things out and in noticing the impact of their new practice.
- readers will engage in dialogue with the articles between issues.
- Mathematics Teaching gives teachers confidence to use ideas which are current and relevant to them in the pressurised world of teaching and learning mathematics.
- articles do not shy away from describing the tensions, imperfections and difficulties that are a part of teaching and learning mathematics.

The focus of articles may be:

- an account of something mathematical.
- an account of a visit to another classroom, another school, or another country.
- a description of using a favourite piece of equipment or resource.
- a reflection on experiences of different teaching and learning styles.
- views and news on current initiatives.
- responses to, or reflections on, articles from previous editions of Mathematics Teaching.

The Editorial Board are:

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**Plus 1** drawn by Harry Venning.
Forewords
by Alistair Bissell.

Welcome to MT265, the last issue under the guidance of the current editorial board. It seems fitting that all of the articles in this issue are critical and highlight the complexity involved in teaching. As Gattegno reminds us, in the powerful and challenging What matters most from MT12, reprinted in this issue’s From the archives, “we have to learn to think in a complex way about complex questions”. While I can understand teachers, with many pressures on them, wanting to find simple and straight-forward ways of teaching, many articles in this issue highlight the dangers of over-simplification. I love the assertion that “teachers must resolve to accept that only they, in their classes, at each instant, will find what to do that is right in the situation”. Classrooms are complex and unique, so over-simplified and generalised approaches are unlikely to be helpful.

I always enjoy reading Danny Brown’s descriptions of working with children on mathematics because they feel so close to the detail of what happens in his own unique context. In this issue, I was intrigued that, despite recognising an efficient way to solve the problem his students were working on, he chose not to share it with them. I have recently been thinking a lot about problem solving and am fascinated by the benefits of not telling students what to do. It feels counter-intuitive, but Danny instead pushes for discussion of a different problem and tries to develop in students a sense of knowing what you are doing and why. Colin Foster also writes about problem solving and teaching students what to do when they do not know what to do, highlighting the issue that what the teacher offers changes the situation that the students are learning about. The idea of encouraging students to apply, in creative ways, the knowledge that they already have, seems like it could leave little room for the role of the teacher, yet finding a middle ground between specific and generic advice may support students in being able to act. While I will pay attention to the detail of what actually happened, I love the sense I have gained, over the last three years as part of the editorial board, that writing about classrooms, paying attention to the detail of what actually happened, allows us to gain conviction about how to act. While CPA may contain powerful ideas, these may get lost if we become constrained by the process and lose sight of what our learners are doing and the meaning that CPA might afford them.

What surprised me was the similarities between the first two articles about problem solving and the next two articles about teaching times tables, where the emphasis, for me, seemed to be on equipping children with the skills to use known facts to generate things that they do not yet know. Again, the subtleties of the teacher not telling seem important in allowing students to generate their own meaning. I am reminded of Caleb Gattegno’s suggestion, in What we owe children, that “memory is a weak power of the mind” and the importance of equipping students with structures and connections to fall back on when they cannot remember, or do not yet know, a given multiplication.

What feels different to the articles about problem-solving, however, is that there seem to be a collection of specific techniques that may be useful for teachers and students to draw upon, whereas Schoenfeld’s questions, that Foster highlights as useful for developing problem solving, seem much more general.

I have been very taken with David Fielker’s article about observing lessons and, in particular, his reporting of two observers disagreeing about their observation of the same student in the same lesson. This, for me, highlights that what we bring to a situation, as observers, can affect our perception of it. After I read this issue’s article from the archive, I found Gattegno’s thoughts infiltrating, and resonating with, my reading of the other articles in this issue. While I am sharing the thoughts that the articles generated in me, I invite readers to find different ideas which resonate with their own experiences.

I believe that Gattegno is giving us permission to challenge the common/conventional wisdom in order to do what we believe is right for our learners. He suggests that our role, as educators, is too important for us not to. Dan Ghica seems to be challenging notions of what might be possible by working with eight-year-old children on knot theory. John Mason and Anne Watson question CPA (Concrete-Pictorial-Abstract) as a mindless process to be followed and clarify some of Jerome Bruner’s ideas which underpin the approach. They warn that the drawing of pictures may not relate to mathematical meaning and question the order. While CPA may contain powerful ideas, these may get lost if we become constrained by the process and lose sight of what our learners are doing and the meaning that CPA might afford them.

I love the sense I have gained, over the last three years as part of the editorial board, that writing about classrooms, paying attention to the detail of what actually happened, allows us to gain conviction about how to act. While I will miss being part of the editorial board, I am very much looking forward to finding out what new aspects the next editorial board will bring to MT.

Several members have contacted MT to point out that another early president of the ATM, Trevor Fletcher, sadly died in April last year. It was remiss of MT not to carry an obituary at the time. If readers would like to share their memories of Trevor we will include these in the next issue as a tribute.

Please email journaleditor@atm.org.uk.
Danny’s diary: Two excerpts


We had been working together for around four months and I thought it would be useful to reflect on what we had done by working on a set of mixed examination questions. I had noticed that J and K often rushed into problems, following first impressions without fully considering what a specific question requires. In the excerpts below, I invited them to articulate what they were doing before writing anything down, in the hope that it might help them comprehend and appreciate what might be required. To this end, I interrupted their thought processes much earlier and more frequently than I might usually have done.

At some point during the lesson, J and K encountered this question:

The roots of the equation \( kx^2 - 3x + 2 = 0 \) are equal. What is the value of \( k \)?

**Me:** What type of question is this?

**K:** Quadratic.

**Me:** A quadratic … what does it bring to mind for you?

**K:** The roots and the quadratic formula.

**Me:** What are you thinking about, J?

**J:** Something to do with the parabola [making a U-shape with his hands]. I thought you could just factorise it, but you’d need \( kx \) and \( x \) in the brackets, which wouldn’t work.

**Me:** OK, so these different actions are available. Can you articulate what you are trying to do?

**J:** We’re trying to find the value of \( k \).

**Me:** What is \( k \), what type of mathematical object is it?

**K:** The value that is multiplying \( x \) squared.

**Me:** Yes, it’s the coefficient of \( x \) squared that you’re looking for … . Is using the quadratic formula going to find what you want?

**K:** No, it’ll give you the roots.

**Me:** J, will what you are going to do give what you want?

**J:** Probably not.

**Me:** All the methods that are coming to mind are methods for finding roots, and this question is asking us to find the coefficient of \( x \) squared.

**K:** We could rearrange for \( k \).

**Me:** Imagine doing that, what’s going to happen?

[J and K start to describe what will happen and realise it is not useful. There is a pause. I invite them to read the question aloud.]

**Me:** OK, what bits of that sentence felt important?

**J:** That the roots are equal.

**Me:** Have we talked about the roots being equal yet?

**J:** No.

**K:** So, we know the graph is sitting on the \( x \)-axis, one point on the \( x \)-axis … . And it crosses the \( y \)-axis at two.

**Me:** That’s a nice image.

**J:** Would completing the square help? No, I don’t think so, you can’t do it with that \( k \).

**Me:** OK.

[Long pause.]

**K:** I give up.

**Me:** OK, you’re stuck. J?

**J:** There must be something we can do … . Could we try thinking of it without the \( k \) for a minute? [J works on solving \( x^2 - 3x + 2 = 0 \) and sketching it] It’s not what you want, now there’s no \( k \).

[Another pause]

**Me:** OK. Shall we come back to this question later?

**J:** Yes, that’s what I’d do in an exam.

**K:** It’s really frustrating when you don’t know the answer.

**Me:** Yes, I know.

I used various probes to encourage J and K to articulate what the question was about and what
actions they might take. They did not associate the roots being equal and the discriminant being zero, which I had perceived to be the most efficient way of coming to an answer and therefore useful in an examination situation. I chose at this point not to give any information about the relationship between the roots and the discriminant or show them how they could have used this to solve this problem. I wanted to see what if it would come back to them after working on related questions.

J and K decided to only work on questions involving quadratics from this point. Later, they came to this question (see figure 1).

J: What type of question is this? A quadratic inequality?

Me: Let’s read it out. [K reads it out]. Can you name anything there?

K: Discriminant.

J: What’s the question about?

Me: About working out what these things do [pointing to the two inequalities], how they look?

K: The discriminant is positive, so it has to be A or B.

Me: Why?

K: Because if the discriminant is negative, then it’s … . But then a is positive as well … . If it was negative it would be upside down … . And then, erm … . If it was that one [points to graph A] then it would be less than zero, or something … .

Me: [Pause] What do you think, J?

J: I think it’s B.

Me: Can you explain why?

J: Because if $b^2 - 4ac$ is greater than zero then there are two real roots, but there is only one there [pointing to the root on graph A]. I’m not sure if I’ve remembered this right, but if the discriminant is greater than zero there are two roots, and if it is equal to zero there is only one root.

Me: And if it’s less than zero?

J and K: No roots.

Me: OK, so that would be a rationale for the answer being B or D.

K: And because $a$ is positive then it must be B.

Me: Right.

I got the sense that J and K wanted to get to the answer and not bother with the discussion, but I felt that the discussion was useful in identifying some confusion and it brought them naturally to the discriminant. Following this, I used the quadratic formula to illustrate how the value of the discriminant determines the number of roots. They then continued to work on more questions. At some point, J stopped what he was doing:

J: Would you use the discriminant in the one that we did before, that one we couldn’t do?

Me: That’s something new to try – let’s go back to it. Can you articulate what you are going to do?
Danny’s diary: Two excerpts

J: The discriminant equals zero, because the roots are equal.

K: But what about the $k$?

J: You can rearrange to find your $k$.

[J substitutes the coefficients into $b^2 - 4ac$, and they rearrange it together, arriving at the correct answer.]

Me: Have you answered the question you’re trying to answer?

J: Well, we have a value for $k$.

K: And I’m certain it’s right. We could substitute these numbers back into $b^2 - 4ac$, but we know it must be zero.

Me: Right, and you have a reason for doing what you did. That feels important, doesn’t it? Not just solving that problem, but…

J: Doing other questions helped us do this one.

Me: Yes, you weren’t able to make the connection between the discriminant and the roots before. When you read “roots are equal” in the future, you might think, “Ah, the discriminant is zero”. Articulating might help you do this, knowing what you are doing and why, which gives some certainty about your answer.

I was excited when J returned to the unsolved problem. The associations that I hoped they would make gradually became available. I wonder if articulating in this way will encourage them to spend a little bit longer comprehending what questions are about and appreciating what is required, allowing them to make relevant associations, rather than rushing to get an/any answer.

Reflection

Reflecting on the first dialogue, I feel that my interventions guided them away from what they were going to do and towards that which I had in mind - the association between equal roots and the discriminant being zero. They might have gone on to find the roots and then equated them, which could have led to a solution and may have been enlightening, albeit less efficient. I pursued this strategy of inviting J and K to articulate throughout the year. There was some evidence that it was useful. J and K certainly became more deliberate when working on mathematics. But I also have some doubts. Anything a teacher says interrupts the learner’s flow of associations. There is a fine line between intervening and interfering. J and K never seemed to want to articulate what they were doing. Whilst it can be helpful to articulate what one is going to do when solving a problem, it can also be a distraction, or not even possible: I’m reminded of Varela, Thompson and Rosch’s (1991) phrase, “The path is often laid down by walking.” (p. 237).

A final note on the mathematics. Some students I have taught over the years find it difficult to make relevant associations between the discriminant and roots when solving problems. I suspect that these students may be trying to hold the relationship as a set of facts to be recalled. This seems to be what is happening in the second dialogue: with K’s confusion, and J’s comment, “I’m not sure if I’ve remembered this right, but…”

More memorable connections may be forged by stressing geometric relationships between the discriminant and the roots. The difference between the roots of a quadratic is $\sqrt{b^2 - 4ac}$. That is, the discriminant is related to the squared distance between the roots. What was a set of algebraic facts to remember is replaced by only one image/idea, which might come, “almost for free” (Mason, 2016).

Danny Brown is at home with his daughter, training as a counsellor and continues to be involved in mathematics education.

References


Thanks to John Mason for his help in writing this diary.
The fundamental problem with teaching problem solving

Colin Foster reflects on dilemmas in teaching students how to solve mathematical problems.

I take problem solving to be “engaging in a task for which the solution method is not known in advance” (NCTM, 2000, p. 52). This is the opposite of doing what you have just been taught how to do, and is more than just applying the method you have been shown to a contextual situation. As many people have said, problem solving is what you do when you do not know what to do. I agree with the mathematician Paul Halmos that solving problems is the principal, pre-eminent activity of the mathematician. When considering what constitutes “the heart of mathematics”, he concluded that this was not theorems and proofs but problem solving: “the mathematician’s main reason for existence is to solve problems […] therefore, what mathematics really consists of is problems and solutions” (Halmos, 1980, p. 519, original emphasis).

But, what about those problem-solving lessons? How do we teach students to solve mathematical problems? If problem solving means not knowing in advance how to solve the problem, it follows that the teacher can kill the problem-solving aspect if they teach, or re-teach, the method immediately before giving students the problem. This may seem like a helpful thing to do in the short term, and it is likely to help the students to solve that particular problem, but it is unlikely to support their long-term ability to solve unfamiliar problems, where hints are not available. I have seen lessons where the teacher has prepared a problem-solving task for the class, but then, just as they introduce the task at the start of the lesson, it is as though they lose their nerve. They say something like, “Just before you start, let me remind you of a couple of things …". And, in so doing, they destroy the problem that they have carefully constructed, and the lesson becomes an exercise in following the teacher’s method. I am sure I have done this myself.

A more subtle way in which the same thing happens is by offering the wrong kind of help to students during the lesson. The teacher is careful not to kill the problem in their lesson introduction by telling the students how to solve it. But, instead, they effectively do the same thing as they circulate around the classroom, making suggestions, giving hints, encouraging the students to take particular approaches. Although this is well meant, I think it can be just as detrimental to students’ learning of how to solve mathematical problems. This raises the question of what the teacher’s role should be in a problem-solving lesson, which is, for me, the fundamental problem with teaching problem solving. If we teach students how to solve the problem, and then give them the problem, it is not problem solving any more, but, if we just give students the problem, it is not teaching any more.

I have seen problem-solving lessons where the rationale seems to be to give the students an interesting problem to solve and then just stand back and let them struggle with it. I think I have taught lessons like that myself, but I am now unconvinced that that is an effective way of teaching students to solve problems. If the students end up solving the problem without help, then I worry that it was not challenging enough. If they do not, then I worry that they have not learned anything.

I have seen problem-solving lessons where the rationale seems to be to give the students an interesting problem to solve and then just stand back and let them struggle with it. I think I have taught lessons like that myself, but I am now unconvinced that that is an effective way of teaching students to solve problems. If the students end up solving the problem without help, then I worry that it was not challenging enough. If they do not, then I worry that they have not learned anything.

How can we make sense of the role of the teacher
in a problem-solving lesson? It seems to me that scaffolding the problem is not what we should be doing. Scaffolding a problem inevitably diminishes the problem-solving aspect. It is a short-term aid to solving that particular problem, but it does not future-proof students in preparation for problems that they have not yet met. What we need to be doing is scaffolding the problem solving, and this is quite different. When we scaffold the problem, we direct students' attention to particular features of the specific problem that they are working on. When we scaffold the problem solving, we give more generic support, which is intended to help students when they meet other similar problems in the future.

If you take this approach, it means focusing your support, and whole-class discussions, on wider aspects of mathematical problem solving than the particular problem in question and it can be hard for students to appreciate what you are doing. Suppose you pose a problem, such as asking students to “Find all the nets of a cube”. They will naturally see this as the objective of the lesson. If you ask a student what they are doing, of course they will say, “I’m trying to find all the nets of a cube.” That is what you told them to do. But, for the teacher, the point of the lesson has to be more than that there are 11 nets of a cube and here they are. That is not an important piece of knowledge, certainly not important enough to spend much lesson time ascertaining. It is not worth remembering. The point of the lesson is to learn something about how to solve that kind of problem. Crucial questions for discussion would include:

• How did you go about it?
• How did you check to see if two nets were duplicates of each other?
• What did you count as a duplicate? A reflection? A rotation?
• How can you tackle the problem systematically, so that you know that you must have found them all and can not have missed any?
• How did you go from “this is how many I found” to “this is how many there can be”?
• What other problems have you met that might be solved in a similar way to this one?
• What problems can you invent that could be solved in a similar way to this one?

These are the generic things that can emerge from working on such a problem. But it is easy for students, and their teachers, to overlook them and fixate on solving the particular problem.

Alan Schoenfeld (1985) offers three generic questions to ask students whenever they are stuck on a problem:

• What exactly are you doing? Can you describe it precisely?
• Why are you doing it? How does it fit into the solution?
• How does it help you? What will you do with the outcome when you obtain it?

Note that there is no mathematical content to these questions. Students need to know mathematical content, of course, but that is not the focus here. The goal of the problem-solving lesson is to apply in creative ways the mathematical knowledge that students already have. During the lesson, and afterwards, students should reflect on what worked and what did not, and why, so that they talk not just about the mathematical content but about their approaches to solving the problem.

In working with Japanese colleagues from the IMPULS project (see note at the end of the article), I have learned about the “Japanese problem-solving lesson”, where it is said that “the lesson begins when the problem is solved”. This means that most of the learning is seen to take place in what Polya (1957, p. 14) called the “looking back” phase, reflecting on the choices made, the paths taken, and the advantages and disadvantages of different approaches. The purpose of time spent solving the problem, or attempting to, is to get students into a position where they have enough relevant experience to contribute to and learn from this critical discussion. I think that that is an interesting way of thinking about the structure of the lesson. The plenary is not an afterthought to tie things together at the end, if there happens to be time. It is the main part of the lesson. What happens before the plenary is seen as preparing students for that discussion.

The great thing about Schoenfeld’s three questions is that if the teacher keeps on using them, lesson after lesson, students begin to get to know the questions. The teacher can ask, “What questions am I going to ask you?” and the students will be able to say them. The questions can be put up on the classroom wall, so the teacher can silently point to them. Eventually, the aim is that students internalise these to support their metacognition. They ask these questions of themselves and this helps them to take control of what they are doing.

However, without the necessary toolbox of fluency with facts, procedures and concepts, students will not find these prompts very useful. This is the difficulty with Polya’s heuristics, such as that if you cannot solve a problem “try to solve first some related problem” (Polya, 1957, p. 114). This is great, provided that you
can tell what a simpler, related problem might be. This is not always easy for a student to do. Making the numbers in a problem smaller does not always make the problem easier. Multiplying by 9 is not easier than multiplying by 10. To take an A-level example, a student trying to solve an integral like \( \int x^2 \sqrt{1 + x^3} \, dx \) might try to use this heuristic by thinking, “This looks complicated. Let’s get rid of the \( x^2 \) at the front and just solve \( \int \sqrt{1 + x^3} \, dx \) first, and when I have worked that out I will try it with the \( x^2 \) in as well.” It seems like a sensible approach. Surely having one factor rather than two to integrate will simplify things? This is a really good problem-solving strategy. But, unfortunately, \( \int \sqrt{1 + x^3} \, dx \) is a much harder integral, whereas if you think about \( \int x^2 \sqrt{1 + x^3} \, dx \) in the right way, as a “reverse-differentiate” problem, the answer can be written down immediately. The difficulty is, how should the student know what is going to be simpler unless they already understand the structure? If they could see what was going to be simpler, then they would not need to simplify it. So, heuristics like “Solve a simpler problem first” may be a good description of what successful problem solvers do, but they are only helpful to students if they have the necessary background knowledge to make use of them.

Polya recognised all of this, of course. He is credited with saying, “In order to solve this differential equation you look at it till a solution occurs to you” (see http://www-history.mcs.st-andrews.ac.uk/Quotations/Polya.html). If we want solutions or solution methods to occur to our students, they need the necessary rich background knowledge to make this to happen. But, it is perfectly possible to have all of the necessary techniques safely inside your toolbox and yet not see how they could help you solve the problem you are tackling. The teacher feels frustrated, because they think that the students ought to be able to solve the problem. They apparently know everything they need to know, but they do not mobilise it in the particular situation they are presented with.

One reason for this may be that the students have met the relevant content only in a narrow range of contexts and have not seen how it might be applied more widely. Another reason may simply be that they have encountered the relevant content too recently. When learning a language, students do not spontaneously and fluently use the vocabulary they have just learned. It needs time to bed in. Similarly, if we want students to make sophisticated use of what they know, it might be better to rely on mathematical content that was learned some time ago and is quite robustly known. Content learned 2 years previously is a rough rule sometimes used at the Shell Centre in Nottingham. This also reduces the tendency for students to assume that the knowledge that they need to solve the problem must be the thing that they have just been taught, thereby switching the task from problem solving to routine exercise. It also acknowledges that if the problem-solving demands are high, other demands, such as procedures and concepts, may need to be lower.

Heuristics like Polya’s can be helpful when students have the necessary prior knowledge to solve the problem but a productive approach is not coming into their minds. Then Schoenfeld’s three questions can be a powerful way to help students to bring that knowledge to bear on the particular problem they are working on. But, the point is not to solve the problem. Much can be learned even when the problem is not solved. The point is to learn something about solving problems.

References


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In May 2018, the joint primary group of the Association of Teachers of Mathematics and the Mathematical Association discussed how to teach multiplication tables well. This was considered a high priority due to the introduction of statutory testing for Year 4 children, in England, from 2020, which has propelled fluent recall of multiplication facts to the top of the agenda. Whilst the group outlined their objections to the proposed test in the assessment consultation, it hopes that this article will provide a useful resource for teachers as they prepare their children. This is considered particularly important by the group as they recognise the importance of children knowing their multiplication facts yet are wary of the way in which the proposed test emphasises rote-learning and rapid recall over understanding of mathematical structures. This article has been written to support teachers to continue to teach multiplication facts in a way that focuses on children's understanding. The group's discussion is shared here and it is hoped it will be a useful talking point and call to action for all those involved in teaching times tables.

Overarching principles

The highest priority identified by the group was the development of conceptual understanding alongside fluent recall of multiplication facts. The group were mindful of the risk rote-learning poses to this, so advocated a thoughtful balance between conceptual understanding and recall-focused activities. Members were supportive of the English national curriculum aims of problem solving, reasoning and fluency regarding multiplication facts and believe that embedding the aims is beneficial to children's conceptual understanding in mathematics. The group identified making connections and spotting patterns as key to conceptual understanding. The following sections illustrate how those might translate to classroom practice and into whole-school policies.

Making connections

For multiplication facts, this means making explicit the link with other operations. For example, can children explain multiplication as repeated addition? Are they confident with deriving division facts from known times tables? Can they solve questions such as \( x \times 6 = 42 \)? Employing representations such as this visual image of the inverse relationship of factors and multiples via triads can be powerful (see figure 1).

![Figure 1: The relationship between factors and multiples.](image)

Similarly, illustrating multiplication as jumps along a number line highlights the link between repeated addition and multiplication. (see figure 2).

![Figure 2: Multiplication as jumps along a number line.](image)

Using calculators can reinforce this link. Cumulatively adding 3 to zero will produce on the screen a sequence of multiples of 3. The ease with which such a pattern can be generated makes it accessible to those children at the early stages of learning multiplication facts. Challenging them with questions such as, "How many times do you need to add 3 to generate 15 on the screen?" before encouraging them to check, is a worthwhile activity. If they can convincingly explain why, we would argue that deeper learning is occurring.

Making links extends to making connections between multiples. At the simplest level, understanding of the commutative law can help those children who struggle with 7 x 3 because they have not memorised multiples of seven. Arrays can be used to help find the answer by changing the calculation to 3 x 7. (see figure 3).

![Figure 3: Array illustrating 3 x 7 and 7 x 3.](image)
In general, thoughtfully planned learning sequences, which encourage children to explore and exploit links, will be more impactful than repeated testing of multiplication facts. A focus on the process, instead of the answer, can be valuable and a good stimulus for discussion. Challenging children to explain to an alien, who does not know times tables, how to work out 6 x 3 is an excellent check of understanding. Other useful challenges might include:

- How would you use 3 x 7 to work out 6 x 7?
- How would you find a number which is both a factor of 64 and 40?

Confidence with links between multiples breeds fluency as it unlocks the potential for known facts to reveal much more. One of the most useful ways of using known facts is to derive new facts. Frequently using sentence starters such as, “If I know ___, then I know ___” encourages children to be creative and seek out their own connections. Flexible, autonomous use of number facts can develop confident learners who are resilient and able to think creatively. For example, if children are unable to recall 7 x 9, they could recall 7 x 10 and subtract 7. Employing Cuisenaire rods, or sticks of interlocking cubes, to represent this provides a physical experience that can become internalised. Creating arrays from squared paper can work similarly. By creating an array for 5 x 8 and folding it in half to show 5 x 4, children can see that 5 x 8 is double 5 x 4. Using squared paper arrays can exemplify the distributive law of multiplication too. After creating a squared paper array for 6 x 9, children can fold to see that this array comprises of three lots of 6 x 3, another derived shortcut.

Equipping children with investigative skills can imbue them with confidence. Arrays which illustrate square numbers demonstrate the aptness of the name. Manipulating them can reveal an elegant pattern which opens a world of possibilities (see figure 4).

Table 1: Pattern spotting table.

<table>
<thead>
<tr>
<th>Multiplication</th>
<th>Table Calculation</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 x 2 = 4</td>
<td></td>
</tr>
<tr>
<td>3 x 3 = 9</td>
<td>2 x 4 = 8</td>
</tr>
<tr>
<td>4 x 4 = 16</td>
<td>3 x 5 = 15</td>
</tr>
<tr>
<td>5 x 5 = 25</td>
<td>4 x 6 = 24</td>
</tr>
</tbody>
</table>

This can be followed by challenging children to find 39 x 41. Whilst this is not within the 12 x 12 known facts range, the question can pique curiosity and exploring elegant solutions to problems can engage and enthuse children.

Connections with real life can also be powerful and there are plenty of examples that marry everyday objects and times tables. Pairs of socks, 5 pence coins, puppy footprints and octopus tentacles provide accessible representations of multiples and children’s own interests should be a driving force. Singing has long been a favourite for teaching tables and familiar songs that count in multiples can be enjoyable and memorable. Focussing on connections removes the risk that children will forget isolated facts. It supports the notion that everyone can do maths because children are being equipped with the skills to use known facts to work out what they do not yet know.

**Spotting patterns**

Pattern spotting is an important element in primary mathematics. It is critical for fluent recall of multiplication facts as it can lighten the memory load. For example, if children know that multiples of 2 have a repeating pattern of 0, 2, 4, 6 and 8 in the ones digit, then it is easy to work out that the next multiple of 2 from 14 is 16. However, it is important to be wary of over-generalisation as children may invent plausible, but incorrect, new rules such as that every odd number is in the x3 table. Challenging them to disprove their conjectures with counter-examples can be a powerful learning experience.

Visual representations of patterns can help secure them in children’s minds. For example, highlighting multiples on a 100 square is an opportunity to help children begin to generalise. Using dials (see figure 5) to join up the ones digits of a times table is another great way to illustrate pattern. Ask children to work out which table the dial in figure 5 shows.
Figure 5: A multiplication dial.

Ask children to identify which different times table could be overlaid in another colour to miss every other digit in this pattern. This simple representation focuses on the ones digit of multiples but using base 10 equipment to represent sequences incorporates the tens which can help highlight other patterns. For example, within the nine times table where the tens digit increases as the ones digit decreases.

Using structured sentences can help “children to communicate their ideas with mathematical precision and clarity” (NCETM, 2015) and “sentence structures often express key conceptual ideas or generalities and provide a framework to embed conceptual knowledge and build understanding” (NCETM, 2015). For example, “3 multiplied by 6 is 18. 3 multiplied by 12 is double 18” is a sentence structure that children could use to highlight relationships between multiplies and derive unknown facts. Here, connections are key and posing the open question, “What’s the same, what’s different between the three times table and the six times table?” requires children to explore and make connections for themselves.

The number line is one of the most important resources to support children in noticing pattern and skip counting along with a counting stick can be a great group activity. Whilst repeatedly saying multiples in sequence helps embed them in memory, moving along a counting stick illustrates repeated addition thus reinforcing the structures of multiplication. Developing this strategy provides a tool for children when faced with a question about a fact that they are unable to immediately recall.

We would also argue that practice is a vital part of learning and that practice that both reinforces procedural fluency and develops conceptual understanding is the most valuable. Practice that elicits and highlights pattern in multiplication facts is helpful. The example in figure 6 illustrates pattern, as well as the importance of the commutative law for decreasing the number of facts that need to be memorised (see figure 6).

Pattern spotting can help nurture enjoyment and curiosity in mathematics and support children’s developing fluency with multiplication facts. Through a focus on pattern spotting, supported by manipulatives and drawings, children develop a sense of number and their sense of whether an answer is right or wrong matures. If they can explain why 17 cannot be a multiple of 3, they are heading in the right direction.

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References


| 1x1 = 1 |
| 1x2 = 2 2x2 = 4 |
| 1x3 = 3 2x3 = 6 3x3 = 9 |
| 1x4 = 4 2x4 = 8 3x4 = 12 4x4 = 16 |
| 1x5 = 5 2x5 = 10 3x5 = 15 4x5 = 20 5x5 = 25 |
| 1x6 = 6 2x6 = 12 3x6 = 18 4x6 = 24 5x6 = 30 6x6 = 36 |
| 1x7 = 7 2x7 = 14 3x7 = 21 4x7 = 28 5x7 = 35 6x7 = 42 7x7 = 49 |
| 1x8 = 8 2x8 = 16 3x8 = 24 4x8 = 32 5x8 = 40 6x8 = 48 7x8 = 56 8x8 = 64 |
| 1x9 = 9 2x9 = 18 3x9 = 27 4x9 = 36 5x9 = 45 6x9 = 54 7x9 = 63 8x9 = 72 9x9 = 81 |
| 1x10 = 10 2x10 = 20 3x10 = 30 4x10 = 40 5x10 = 50 6x10 = 60 7x10 = 70 8x10 = 80 9x10 = 90 10x10 = 100 |
| 1x11 = 11 2x11 = 22 3x11 = 33 4x11 = 44 5x11 = 55 6x11 = 66 7x11 = 77 8x11 = 88 9x11 = 99 10x11 = 110 11x11 = 121 |
| 1x12 = 12 2x12 = 24 3x12 = 36 4x12 = 48 5x12 = 60 6x12 = 72 7x12 = 84 8x12 = 96 9x12 = 108 10x12 = 120 11x12 = 132 12x12 = 144 |

Figure 6: Patterns of multiplication facts.
Learning times-tables facts versus learning about times tables

Caroline Rickard and Lorna Earle explore young learners’ knowledge of multiplication facts and multiplicative structures.

The learning of the multiplication facts up to 12 x 12 is currently in the spotlight in England, with the government introducing a times-tables check for all 9-10-year-old children from 2020. Testing takes a black and white stance; either you know each separate fact or you do not. Or, in the case of the timed tests, perhaps you do know it but there is not enough time to submit your answer. In MT201, Victoria Williamson describes incidents that “…illustrated for me that the children perceived the multiplication tables as a series of facts which were either known or not known, not as tools which could be used to derive other facts in a flexible manner.”

As mathematics educators who view times tables as a connected body of knowledge, with some known facts acting as a springboard or starting point for generating others, we were keen to find out whether children recognised connections between facts and knew shortcuts to working out any they did not know. We believe a deeper understanding can result in gradual acquisition of more facts.

We would agree that it is useful to have a stock of known facts. However, some children learn their times tables with ease, maybe suggesting good memory, whilst others do not. Searching for times-tables activities online leads predominantly to activities designed to practise known facts, such as songs to chant along with, or games to test knowledge of multiplication facts. The emphasis here seems to be upon repetition and memorisation. Far fewer activities, located online, seem to offer support for understanding the facts or for strengthening relationships between facts. Rather like learning spellings, we felt that a more investigative approach might be beneficial and that resources such as Numicon could play a suitable role.

Working from the principle that connections between times-tables facts are helpful for children, we noted certain skills we would want them to possess or to develop:

• An awareness of the commutative law: I might not know my 7s for 7 x 5 but realise I know the answer to 5 x 7 as I know my 5s and I appreciate the answer will be the same.

• The ability to use nearby known facts: For example, I know that 7 x 7 = 49 therefore 7 x 8 must be 56 as it will be an extra 7. This could involve going up or down to get to the required multiple.

• The ability to scale answers up by doubling: For example, that multiples of 4 are double the multiple of 2, and the multiples of 8 double the multiples of 4.

• The ability to scale answers down by halving: In particular, that if I know a multiple of 10 then I can calculate the multiple of 5 by halving.

• The ability to partition where appropriate: Useful for multiplication involving numbers greater than 10.

• Developing awareness of the odd or evenness of an answer. This can help us to spot an unexpected answer and allow us to check it.

We approached two local primary schools and gained permission to undertake our research in them, with children aged between 8 and 10. We interviewed the class teacher and year-group coordinator. Both described teaching that focused on the concept of multiplication as opposed to the teaching of times tables per se. The teachers each selected six pupils who they felt would benefit from being involved in the research. None of the children selected to take part in the research had rapid recall of all of the times tables, but they all knew some facts and so had knowledge to build upon. The children were interviewed individually at the beginning of the project. We asked them, “What are times tables?”; “Are you good at them?”; and “How do you learn them at school and at home?” The children described various ways in which they learnt or practised their times tables at home, generally relying on repetition strategies or tricks. They all described, with clarity, how they were tested on their times tables in school.

We looked closely at whether the children used efficient approaches when completing ten carefully chosen times tables questions (see figure 1). Most of the children demonstrated good awareness of the commutative law. For example, a child tackling 8 x
Five saying, “I’m not strong on 8s so I’d swap it round”. Other efficient approaches were demonstrated. For example, a child who knew 8 x 10 would be 80 and who counted back 8 to get to 72 (8 x 9). In contrast, an emphasis on counting on in multiples of whatever was typical, often from the start and sometimes counting in ones to work out the next multiple. Those who relied heavily on counting were prone to making errors.

We also noted whether any explicit links were made between the questions. This was occasionally the case. For example, one child choosing to follow 4 x 5 with 8 x 5 noting it is “a bit the same as it’s double as 4 add 4 is 8” and another tackling them in the opposite order, saying 4 x 5 was going to be 20 because it is half of 40 (8 x 5).

After the initial interviews, we put together a collection of activities we could use over a period of four weeks. These were designed to help children develop deeper understanding and more efficient strategies, as opposed to focusing on remembering more facts. The idea of efficiency was explored at various points with the children, relating this to the idea of being a “lazy mathematician” and not wanting to do any more work than is necessary. An idea the children took to. Concrete resources and visual images played a part as did discussion, in order to emphasise reasoning and to draw children’s attention to possible connections between multiplication facts. At times, the distinction between multiplication and division was deliberately blurred, with understanding of one seen as integral to the other. Whilst we used the same materials, one of us introduced ideas to the whole class and then worked with a group of six children (three 8-9-year-olds and three 9-10-year-olds), the other worked with six 9-10-year-old children drawn from across the year group’s three classes.

In our short time with the children, we saw them respond positively to the strategies, making connections between them and developing and applying them to new scenarios, although sometimes needing prompting to do this.

In the first session, the children explored jumping to a times-table fact they knew and working out related ones from there. For example, knowing 6 x 5 = 30 and adding on 6 to find 6 x 6, rather than counting in 6s from the start (see figure 2). Most of the children gave examples that were either one above or one below the given times-table fact. Some did identify that they could also double or halve the given fact to generate other new facts.

The idea of using a known fact was evident in some of the children’s work on arrays the following week. Splitting arrays in ways that identified known multiplication facts meant answers could then be combined (see figure 3). This was modelled to the children before they tried some independently, essentially investigating the distributive law. It was evident from their work and conversations that many of the children were using the ideas from the previous week to help inform their splits, and one child even independently checked their answer to 8 x 9 by working out (8 x 10) – 8.

In the third session, the children were encouraged to find times tables “hidden” within each other. For example, the 4 times table in the 8 times table. This was explored using both number grids (see figure 3) and visual images like arrays.
4) to prompt discussion, and *Numicon* to show how two 4-pieces fit into each 8-piece (see figure 5). These ideas were also used to explore odd and even number patterns within times tables, providing a visual representation of the odd-even nature of odd times tables and why even times tables will only have even numbers in them.

![Figure 4: Interactive teaching programs number grid.](image)

![Figure 5: Using Numicon.](image)

Having explored this in the third session, the children confidently investigated how different numbers could be made by multiplication in the final session, again using *Numicon* to model ideas. For example, if $6 \times 7 = 42$, then each 6-piece of *Numicon* could be exchanged for two 3-pieces so $3 \times 14$ would also equal 42. Working independently, one child showed a firm grasp of these ideas. Having laid out ten 10-pieces of *Numicon* she realised she did not have to try and find lots of 5-pieces as she could just count up in twos to find that $5 \times 20 = 100$.

In the example shown in figure 6, two of the children worked together to find the factor pairs of 60. We sometimes offered suggestions for pieces of *Numicon* that could be considered, but then left the pupils to work out how many of each piece they would need. As the annotations show, the pupils were able to describe connections between the factor pairs and use these to develop their ideas.

![Figure 6: Factor pairs of 60.](image)

After the sessions, we conducted a final interview with the children with some multiplication-based questions to consider. We saw a number of examples of children reverting back to their reliance on counting on to answer unfamiliar questions, or because, “I don’t like guessing, I like making sure”. When prompted, however, many realised they could use more efficient strategies. This confirmed our belief that it is beneficial to embed efficient strategies and reasoning about times tables as early as possible. Resources, such as *Numicon*, and activities focusing on mathematical reasoning could usefully be explored with children as they learn the 5 and 10 times tables, embedding the ideas of links that can be further explored when learning the 2s, 4s and 8s or the 3s and 6s. Strategies such as working from a known fact (for example, I know $5 \times 10 = 50$, so $5 \times 9 = 45$) could be encouraged early on, and then used explicitly, when teaching the 9 times table for example. Small details such as agreeing a consistent approach to how calculations are presented if you are trying to signal a particular times table e.g., $5 \times 1, 5 \times 2, 5 \times 3$ and so on, or $1 \times 5, 2 \times 5, 3 \times 5$, may also help the children apply strategies confidently.

Activities that encouraged mathematical reasoning, such as responding to true / false statements,
Learning times-tables facts versus learning about times tables

the children an opportunity to consider and explain their understanding of concepts. They rehearsed and developed their explanations of the patterns and generalisations they were spotting. In the example shown in figure 7, taken from the final interview, the pupil began by counting on from a known fact. She then provided a justification that only in the one times table would you get answers that were consecutive. When questioned as to whether there could be an easier way to prove it, she suddenly exclaimed, “Ah, it’s an odd number! If it’s an even times table the answer will always be even.” She then wrote, “Mr. _ is wrong because 83 is an odd number”. Although if she had read the question more carefully, she would have realised she had in fact proved the teacher correct (see figure 7).

Developing a culture of reasoning may help shift children’s mindsets from times tables being a collection of isolated facts to be learnt, and to be tested on, to them being an intriguing aspect of mathematics, full of interesting connections to notice and explore. This was exemplified by one child, who, when shown a Venn diagram showing multiples of 3 and 6 (see figure 8) said, quickly, “But, there won’t be any numbers in there” [pointing to the section on the right]. He went on to explain, “There are two threes in every six,” modelling this idea for the others in the group with the Numicon they had been using. The following week, he was shown the same Venn diagram again, and he said, “I think three-quarters of the numbers would go in just 3s and one quarter in the middle.” We discussed this, coming to the conclusion that for this Venn diagram half the numbers would go in each circle, but if the criteria had multiples of three and the multiples of 12, his split would be correct. This exchange highlights how he continued to consider the ideas he had been introduced to and how further reasoning opportunities helped to develop them (see figure 8).

We felt that our exploration of multiplication facts benefitted from the relaxed environment we were working in, avoiding any focus on speed and recall. One child explained in the initial interview that he enjoys looking for patterns in times-tables, but that these patterns often desert him in the tests. After the taught sessions two of the children explained, unprompted, that they had begun to use the strategies when completing their weekly times tables tests. One of them was proud that he had finally completed the multiplication grid he was working on. He explained that he thought the sessions had helped achieve this as, “We’ve gone over lots and lots of times tables and worked out quicker ways [of doing them],” before adding, “It’s really helped. I would want to keep doing this.”

At the end of the final interviews some of the children engaged in further conversations about the purpose of the sessions and whether they had found them helpful. When discussing the idea that the sessions had not been designed to build a collection of facts one child summarised, “It’s been about knowing methods.” He went on to explain that whilst, “I feel about the same confident with my times tables [facts] but know more methods so find it much easier now.” Another child summarised it slightly differently, excitedly declaring he could work out times tables by being a lazy mathematician.

In looking at everything the children produced we have become increasingly convinced that there is a subtle but important difference between learning times-tables facts and learning about times tables.

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Algorithms . . . Alcatraz: Are children prisoners of process?

Ray Huntley and Chris Hurst describe a study that explored the extent to which students resort to written algorithms for computation.

Multiplicative thinking is a key component of mathematics that underpins much of the mathematics needed beyond middle primary years. We define multiplicative thinking as:

- A capacity to work flexibly and efficiently with an extended range of numbers, for example, larger whole numbers, decimals, common fractions, ratio and percentages.
- An ability to recognise and solve a range of problems involving multiplication or division including direct and indirect proportion.
- Having the means to communicate this effectively in a variety of ways, for example, materials, words, diagrams, symbolic expressions and written algorithms.

The evidence presented from this relatively small sample of students suggests that this flexibility may be compromised by the preference for using algorithms. A number of students demonstrated sound understanding of the mathematics that underpins the multiplication algorithm and therefore how and why it works. For instance, students were able to say that the zero was placed in the second line, because they were multiplying by tens, not ones. Perhaps of more concern is the range of procedures adopted by a number of students for dealing with zeros and the decimal point when multiplying and dividing. Some students were able to explain that “adding a zero” was the same as multiplying by ten, but this was not the case with all students.

We believe that it is important for students to understand the mathematics that underpins algorithms, rules and procedures and that the mathematics should be explicitly taught. Also, it is important for algorithms to be used as one way of calculating along with a wide range of mental and other strategies. If this is done, it might be possible for students to avoid becoming prisoners of process. One point we want to make here is that we have no objection *per se* to the teaching and use of algorithms such as those described. Our main concern is that students opt to use algorithms when there is no need to do so and they are often used at the expense of mental computation strategies.

Two data gathering instruments, a written multiplicative thinking quiz (MTQ) and a semi-structured interview, were used in this study involving two primary school classes at a school in Plymouth. The quiz was administered to both classes on the same day under identical conditions. In the MTQ, students were asked 18 questions. Several questions specifically asked them to show how they would calculate an answer. We wanted to find out the extent to which students employed a learned procedure or rule, or used the written multiplication algorithm in answering the quiz questions. All students were interviewed within four days of completing the written quiz. They were asked questions that specifically required them to show how they worked out a range of calculations. In addition, we asked probing questions on the basis of their responses to some of the questions. This article focuses on responses to the interview questions.

During the interview, students were asked several questions that were designed to ascertain the extent to which they used an algorithm and partitioning, and how they could explain their thinking. The use of the interview questions in this way was purposeful in that we wanted to find out the extent to which an algorithm was used, whether it was used as a convenience and/or in situations where it might not have been the best choice of computation method. The interview questions relevant to this article were:

- 6 x 17 and associated extended facts such as 60 x 170 and 1700 x 6.
- 89 x 3 the question from the MTQ, with variations for probing.
- 23 x 400 and associated examples such as 2300 x 4 and 2.3 x 4.
- 29 x 37.
- 200 ÷ 13.

Five general observations were able to be made from the interviews with considerable consistency shown across the cohort of 56 students. First, most students chose to use an algorithm to calculate answers for number facts derived from 6 x 17, even when they
knew that the answer to that was 102. The following sample from student Johnnie (figure 1) is typical of what a number of students did during the interview.

There are two points of interest in Johnnie’s work. First, he had already worked out the answer to $17 \times 6$ but persisted with using an algorithm to calculate the extended fact $170 \times 6$. He did the same when asked to work out the answer to $1700 \times 6$. Many of the students who were interviewed used the algorithm as a first preference for calculating 1-digit by 2-digit multiplication examples. He also needed to calculate the two parts of the partition and did not understand or trust the distributive property, even though the answer to $17 \times 6$ had already been established from a previous question about the commutative property.

The second observation, also illustrated in figure 1, pertains to the partitioning and the distributive property. Once the distributive property was established, that $89 \times 3 = (80 \times 3) + (9 \times 3)$, I asked, “What would give the same answer as $(20 \times 7) + (3 \times 7)$?” Many students were unable to make the connection and wanted to calculate the two answers and add them together. The sample from student Brandon (figure 2) is indicative of this.

Brandon used the vertical algorithm to obtain the correct answer but said that he did not know if the partitioned example would give the same answer, so he worked out each part of the partition, again with a vertical algorithm for $20 \times 7$.

Thirdly, most students used a vertical algorithm to calculate $23 \times 400$, with many of them writing the larger number on top. Many did that even after calculating $23 \times 4 = 92$. They persisted in using the algorithm when variations such as $2300 \times 4$ and $2.3 \times 4$ were given. Samples from student Calvin (figure 3) are indicative of this.

Fourthly, almost all students used the standard two-line vertical algorithm to calculate the answer for $29 \times 37$, though one used a four-line algorithm. Many also placed the 37 on top as it was the larger number. It was interesting to note that the first thing most students did was write their working as in figure 4, before beginning the actual calculation. When asked about this, student David responded in this way:

Well, there are two numbers [indicated the 29 on the second line] so there will be two lines of working, and I put the answer underneath in those two lines. The zero goes there because this column is multiplying by tens.
Figure 4: Typical setting out of algorithm before doing any calculation.

All students who were asked to work out $29 \times 37$ used the vertical algorithm as shown in student Ben’s sample (figure 5). Some put the larger number on the first line and when asked why, were only able to offer an explanation like, “Because that’s what I usually do”. The way in which Ben has shown the numbers that were “carried” is reflected in the work of the other students.

Figure 5: Sample from student Ben showing typical setting out of algorithm.

The fifth general observation that can be made pertains to the use of a division algorithm for $200 \div 13$. Several students referred to this as the “bus-stop method”, one describing the reason for this being that, “It is like a bus shelter that covers the numbers you are dividing into”. All but one of the students who were asked to work out the answer for this question began by writing the algorithm and then immediately wrote answers to “the thirteen times table” at the side. No student in the sample attempted to work out the answer in any way other than using the algorithm. A sample from student Ian is shown in figure 6. To follow up the $200 \div 13$ example, he was questioned as to how he would do $200 \div 3$. It is interesting that he used the same procedure when it would be reasonable to expect that he would not need an algorithm but would work it out mentally.

Figure 6: Sample from student Ian.

As well as these five general observations, some responses from individual students need to be noted as they demonstrate a range of other ways in which students rely on procedures and rules for arriving at answers. Such rules and procedures do not necessarily reflect a real understanding of the mathematics but rather they constitute “rules without reasons”, to use Skemp’s (1976) term. However, as is shown in some of the following samples, there may be some sort of accompanying partial understanding or even indeed quite a reasonable level of understanding, which is overridden by the application of a rule.

Student Caitlin drew a place-value chart to demonstrate $74 \times 10$ and described moving the figures “across one place to the left”. When asked about the zero that appeared in the ones column, she called it a “bouncer”, as it is there to stop the other figures getting back into that column, but she also then used the word “placeholder”. When she was then asked to calculate $3.6 \times 10$, she extended her place-value diagram to show tenths and hundredths including, despite labelling them “Th” and “h” and including a thinner column for the decimal point:

C: I go 3, then that’s the column for the decimal point, and then you put the 6 on this side, and then you move it across … .

Interviewer: So the decimal point has a column?

C: Yes so it makes 3.6.

I: So the 3 goes here [in the ones] to make it 10 times bigger, and the 6 goes here [in the decimal point column]?"

C: No, it’s meant to go across the column.

I: So, it jumped across this column?

C: Yes.
I: So why can’t it go in this column [decimal point]?

C: Because it’s a fence . . . you have to go round it, you can’t go in it. I just made that up.

Student Franklin demonstrated an interesting combination of understanding and procedures as is shown in the following exchange. He had written the correct answer for $2.3 \times 4$:

F: It’s the same basically. You just cover over the decimal [used his finger to do that] and you can see the sum as 23 times 4. You pull your finger away and there’s the decimal point so you put it in there [indicated between the 9 and the 2] to make one number after the decimal.

I: When you put your finger over the decimal point in the 2.3, what are you actually doing to the value of the 2.3?

F: You’re timesing it by ten.

I: Brilliant. So to make the answer correct . . .

F: You have to divide by 10.

Student Robbie also demonstrated understanding of how digits moved one place each time a number is multiplied or divided by a power of ten. However, in the following sample and interview he opted to use algorithms and talked about “the number of zeros behind the decimal point” instead of employing his understanding of digit movement:

R: $[24 \times 0.06]$ I would divide it [the 24] by 100 so it would be 0.24 by 6.

I: Why would you divide it by 100?*

R: Because the equivalent of that [0.06] is 6 divided by 100. [He set out a vertical algorithm.] You’ve got to remember that there are two numbers behind the decimal point so it is 1.44.

Figure 7: Samples from student Robbie.

Some students regarded a zero as something to be manipulated and during the interviews it was very common to hear students talk about “adding a zero” or “taking off a zero”. The following comment from student Zachary is interesting, given that he had earlier described digits in a number moving to a different place when multiplied by ten.

Z: [When given $23 \times 400$, he wrote the vertical algorithm for $23 \times 4$ and arrived at 92.]

I did 23 times 4 and you get rid of the zeros because they’re not doing much to the sum, so you just take them off. 23 times 4 is 92 and so because the zeros are there, you have to add them back on and it becomes 92 thousand.

[He corrected it to 9200.]”

Earlier in the interview, Zachary had given the following answers mentally: $70 \times 5 = 350$, $700 \times 5 = 3500$, and $70 \times 50 = 3500$, yet he used an algorithm to work out the answer for $350 + 7$.

Reflections

Multiplicative thinking is a core aspect of learning in mathematics, and includes many linked components, such as arrays, multiples, factors, the times-bigger relationship and mental and written calculation methods for multiplication and division. In this study, it is apparent that across schools on both sides of the world, teaching tends to be focussed on multiplication and division as separate processes, tentatively linked as being “inverse operations” at some point. We contend that the operations should be taught together as they are two aspects of the same idea. The responses from the students give insight into how this teaching approach restricts students from developing this overarching idea, and discussion with teachers in different schools highlights that they are more concerned with students being able to answer particular types of question in national testing, again in both the UK and Australia. The removal of such testing (as advocated by MT editor Tony Cotton in MT259, p. 42) would perhaps free up teachers to develop their own subject and pedagogic knowledge to be able to teach multiplicative thinking in a deeper and more relational way.

Reference


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What is the P ... in CPA?

Anne Watson and John Mason reflect on the interpretation of the ‘P’ in the concrete/pictorial/abstract (CPA) approach.

Consider the elementary task of shading one tenth of each of the shapes below (perhaps seen as chocolate bars?)

Figures 1 and 2: Chocolate bar portions.

Now shade in one tenth of the interval from 0 to 1 on the number line.

Figure 3: Unit interval in tenths.

In the first two, any square will do; on the number line there is ambiguity, because while shading any segment indicates one-tenth of the whole, shading the first segment from the left indicates a measure of one-tenth, and marking the end of the first segment displays 1/10 as a position on the number line.

While the first two appear to be inviting actions on diagrams, they depend on both previous experience of subdividing and colouring in and at least a sense of a generalisation concerning the action of subdividing a whole into an appropriate number of equal parts and choosing one of those parts. The number line task provides a diagram, but this carries with it a convention that 1/10 is the label for the right-hand end of the first of the ten equal segments into which the unit interval has been subdivided. We ask ourselves, what has this to do with the P of CPA?

“P” is used by Andy di Sessa (1987) to provide a shorthand, “p-prims”, for “phenomenological primitives”. These are the intuitive understandings that arise from past experience, usually physical situations. For example, “multiplication makes bigger” is readily extracted, or abstracted, from early experience of multiplication, and “you cannot take larger from smaller” applies only for a limited time, until negative numbers are introduced. Rather than p-prims being treated as major misconceptions that ought to be cured or avoided, they can be treated as temporary states during the construction of conventional knowledge. The P stands for what arises from experience and alerts educators to think about what obviously arises from a particular experience. If this is an incorrect mathematical generalisation then other experiences that contradict it are needed.

In the first two shading tasks, the p-prim associated with shading any square could be “because one tenth means one of ten equal parts” or “because the chocolate bar can be shared equally between ten people”. On the number line, a possible p-prim could be a journey from 0 to 1, or one of ten equal steps in such a journey. It is also worth noting that the tenths in the first two are congruent as well as equal, so an associated p-prim, that may need some contradiction, might be based in congruence rather than in equal quantity.

Anticipating p-prims can help with your choices of pedagogical examples. When working with tenths, of fundamental importance in connecting measures, decimals and fractions, what variations of materials, diagrams, language and experiences do you offer so that learners develop a full sense of tenths?

Behr, Lesh, Post and Silver (1983) point out that a teacher may illustrate a mathematical idea using various concrete manipulatives but that this does not ensure a strong connection between learners with the same materials and their understanding of the mathematical idea. Copying actions does not ensure cognition. Shading squares that have been counted is one kind of action; explaining why this indicates a particular fraction requires a sense of structure, the use of language forms and the use of numeric and fraction symbolic norms; knowing when any segment will do and when it has to be in a particular place relates to the use of diagrams.

Furthermore, in their extensive research on rational numbers, they found that the choice of concrete materials in teaching fractions, and hence the choice of what is ‘pictured’, is complex. Firstly, different materials prompt different intuitive understandings, invoking and provoking different p-prims. They used
counters, egg boxes, lengths, areas, rods, number lines, folded paper, tilings and graphs. These were needed because rational numbers have several different meanings: parts of a whole, quotients, ratios, measures, numbers and scalings. Each of these required different materials with correspondingly different actions. Secondly, materials that create some confusions to be resolved seemed to be more effective for learning than materials that present no problems. Shading one tenth of a chocolate bar that has ten equal but non-congruent pieces would invoke different p-prims than one based on squares.

The complexities of using materials, pictures and diagrams can be seen in the following example of sharing food when the sharing is problematic. How might three chocolate brownies be equally shared between five people? It can be done in various ways with various potential mathematical possibilities: minimising cuts; and demanding congruent portions. Here the brownies themselves are concrete, and the apparatus is also concrete, but the brownies are also “real” while the apparatus is a representational structure, that is, somewhere on a continuum between real objects and symbolic representation. The ultimate aim could be to understand that quotients might be expressed as fractions.

Choice of manipulables for this problem is important. For example, representing each brownie with a Numicon 5, as some people do, suggests that the problem has already been solved, since each brownie has already been imagined as having five equal parts. But if the problem is now varied because an extra person arrives, each Numicon 5 would have to be replaced by a Numicon 6. This being a different size is not a helpful representation since real brownies do not change in size. The Numicon is an abstraction.

There is confusion here between using a diagram to solve a problem and using pictures to communicate a solution or a way of thinking. The materials need to have the same potential for variation as the real experience. In this problem, the total amount of cake is invariant. Either the number of people and hence the size of each serving vary, or the size of each portion and hence the number of people vary. Both the concrete materials and the diagrammatic representation have to have these qualities. A good diagram can itself be manipulated and retain some mathematical meaning. For example, movement along a number line can be converted into numbers, or into generalisations about such movements, or into actions with real objects, or into imaginary situations.

Diagrams, and some manipulatives, carry with them conventions about how they can be used so as to retain mathematical meaning.

We have deliberately moved away from Andy diSessa’s use of “P” to the now ubiquitous (in the UK) use of “P” for “pictorial”. Understanding concrete-pictorial-abstract (CPA) as three consecutive stages of work does not work in the brownie situation. Concrete could apply to the brownies or to some manipulables used to represent them. Pictorial suggests drawing either of these, when what is most helpful is a diagram that focuses attention on the key relationship(s).

In some cases, a drawing of manipulables might serve as a diagram, but in others it might have extraneous features that are unnecessary for the mathematics, even to the point of being misleading. For example, an assiduous child might want to draw rods as three-dimensional objects, or decorate the pictured brownies with marshmallows. We have seen 11-year-old children drawing highly decorated ice-cream cones to demonstrate a “pictorial” understanding of division as sharing. We have heard about children being asked to draw pictures of the apparatus they have been using in mathematics lessons. This needs some critical thought. For example, on our kitchen table we keep a bag of over 100 small plastic brown bears. Children who visit us often sort these into various groups spontaneously, constructing patterns, sets and so on (see, for example, figure 4) with nearly all such sortings offering potential mathematisation.

Figure 4: Teddies organised onto a chessboard waiting to be sorted.

Our nightmare vision of the use of P to mean “draw a picture” dooms small children to spend hours drawing bears, an activity with little mathematical merit as far as we can see, however much they may enjoy it, or hate it. Drawing pictures of your mathematical work is not a useful interpretation of P because a picture can
What is the P ... in CPA?

trap children into attending to all the features of the material situation, whether mathematically necessary or not, where a diagram can focus attention on those features that are mathematically necessary and related and how they can be manipulated. When working mathematically, it is not immediately clear what is necessary and related. There has to be some to-ing and fro-ing between the situation and its representations and it is in the to-ing and fro-ing that sense is made.

John Dewey (1933) observed that experiences for children are not concrete simply because teddies or counters or dots are used. They are tools for the construction of meaning. Deborah Ball (1992 p. 47) reminds us that “although kinaesthetic experience can enhance perception and thinking, understanding does not travel through the finger tips and up the arm”. Kath Hart (1993) repeatedly pointed to the pedagogical gap between getting children to use manipulatives and expecting them to have made mathematical sense of what they were doing. These insights, repeated over decades, seem to be forgotten in some uses of the mantra “CPA”.

So why has it even been suggested as a useful idea? The modern origins of CPA lie in the three modes of (re)presentation identified by Jerome Bruner (1966): enactive, iconic and symbolic (although Plato got there first). For Bruner, enactive (re)presentations were material objects, whether found in the child’s world or introduced as teaching apparatus. This aligns with the psychological use of “enactive” to refer to physical behaviour in the material world. Bruner used “iconic” to refer to images, which could be mental images, diagrams or pictures that present the material objects, being readily identified by anyone immersed in the culture. These provide a steady backdrop on which to imagine actions. Bruner treated conventions and anything that required interpretation or instruction in order to know what it referred to, as “symbolic”.

In mathematics it is more helpful to see enactive elements as any object that can be confidently manipulated, of whatever form. So, numerals are initially symbolic for very young children, but often move through an iconic mode to become symbols, which are then confidently manipulable in the performing of arithmetic operations. Mathematics is notable for its layer upon layer of the symbolic becoming enactive as learners become more familiar with symbols and their uses. It is even more helpful to think in terms of a spiral that can be traversed forwards and backwards, where manipulation leads to a sense of underpinning relationships that are expressed in various modes, including words, which can become manipulable entities in their own right.

Seeing CPA as a three-stage sequence is a degenerate form of Bruner’s insight, especially within mathematics. Even in elementary mathematics this reduction does not make sense, as we have illustrated above. A classroom tool was developed along more complex lines by Wong (2009) in the form of a think-board (see figure 5). Children have to make use of six different presentations of an idea or a problem, but filling this in without learners making links between the modes for themselves can also become a meaningless ‘copy and complete’ task.

Figure 5: Wong’s think-board.

Behr et al. (1983) also found that they needed to use a complex version of Bruner’s insights when teaching rather than a simplified version, and their model makes it clear that it is the transformations between modes that enable learning rather than the modes themselves. In figure 6 we have adapted their version by separating pictures from diagrams. In their version, pictures and diagrams were seen as one mode, possibly because they had not seen the kind of misunderstandings about the role of pictures that we are seeing. We have retained their other features: complexity; multiple connections; multiple directions; and the emphasis on transition between modes.

Figure 6: Modes of presentation, adapted from Behr et al.
All possible modes can be related to each other, either directly or via intermediaries, by using various routes between them. What is important in this model, and what is used in their research, are the two-way transitions between modes and the avoidance of simplistic assumptions about order.

We have used this complex model to think about a Cuisenaire-rod task used by Caleb Gattegno in his classroom film in Montreal (see also Williams, Gregg & Ollerton, 2018, p. 11). Hold two randomly chosen Cuisenaire rods of different lengths behind your back and describe the relationship between them. The material and manipulative modes are the same in this task. To turn these experiences into descriptions of ratio or of fractions takes some hard work. You cannot see what you hold, so you cannot draw a direct picture. You have to imagine it in your mind’s-eye. No teacher can show you what to draw to express your experience. What you might draw is likely to have some diagrammatic qualities but is possibly approximate and might not help you to be precise. You might not even want to draw anything, because of the guesswork involved. You might hear other people describing what they think they have and how they thought about it. This might send you back into an enactive mode while you do more fiddling and more thinking. Words inform your actions as well as arise from actions. Eventually you might have a form of words that enables you to draw a good diagram and/or write a symbolic version of the relationship that is behind your back. You might also have developed new awarenesses about comparing lengths, such as the role of difference, or the effect of iterating a length. Learning is complex, “CPA” cannot capture either the task or the learning experience. Its use is as an aide-memoire for teachers, not as a routine for children.

References


Tables together

Activities taken from the ATM publication, Tables together: A complete programme for learning multiplication tables at KS2.

After establishing up to, say, 6, 12, 18 it is worth going back to 1.

Move down from 1, 2, 3 then along the three row; 3, 6, 9, 12, 15, 18 …

You can also use the slide at this stage to recite the table in its standard way but still using the slide as a visual image, saying four (tap on 4 in the top row) times three (tap on $\times$3 at the beginning of the three row) equals twelve (tap the square under 4 in the three row). This is about transferring the established image and linking it to the standard way that children say their tables.

Comment from a teacher in the trial.

The vast majority of my class were already secure in their three times table so I used the larger numbers in the next two slides. This was useful and something I would use in future with upper KS2. Times three and doubling is a method some of my class use already. Using the slides with bigger numbers (slides 6.2 and 6.3) provided a good mental workout.

What is the first number in the 70s in the three times table?

I know that 72
18 in the 3x table
because 20x3 = 60
3x3 = 9 and if you
add 12, it will be 69 and add
another 3 will be 72. 3x3 = 9
10x3 = 30
60
60
2x3 = 6
70
A ROUTE TO THE THREE TIMES TABLE DOUBLING AND ADDING

In some of the classes in the trial schools a good number of the Year 4 children were already confident in their three times table. Even for these confident children, it was an insight that there is a connection between the two and the three times table (as in slide 3.0). All of the Year 3 and 4 teachers in the trial schools worked for a whole term on the two, four, five and ten times tables before using the three times table slides.

Slide 3.1

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>×2</td>
<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>×3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Start by establishing the ×2, ×3 table links and patterns.

To do this you need to work down from the left hand side:

Tap 1 chant ‘one’, tap 2, (the square immediately underneath one), chant ‘two’, move the pointer back up to 1, don’t tap but slide the pointer down from 1 to 3 the square under one on the third row, tap, chant ‘three’.

Next; Tap 2, chant ‘two’, tap 4, chant ‘four’. Now slide from 2 to 6, tap six, chant ‘six’.

Tap 3, chant ‘three’, tap 6, chant ‘six’. Next slide from 3 to 9. Tap nine, chant ‘nine’ . . .

Next; Tap 2, tap 4, now slide from 2 to 6, tap six.

Tap 3, tap 6, next slide from 3 to 9. Tap nine . . .

It is worth taking time on this, working from one via two to three, then from two via four to six. And so on. Even those children who know their three times table, and there will be some in each class, can benefit from getting familiar with the idea of doubling, holding the two numbers, one written, one remembered, in their head and adding them as a technique for multiplying larger numbers by three.

Move slowly from the just known to the nearly known.

Tables together: A complete programme for learning multiplication tables at KS2 is available from the ATM at https://www.atm.org.uk/shop/act114pk.
A knot theory for eight-year-olds: Part 2

This is the second in the series of pieces by Dan Ghica in which he describes how he introduced young learners to complex algebra.

You will have read in the last issue that these articles reflect on the process of running a mathematics club for 8-year-old children, in which we are reinventing (an) algebraic knot theory. We are not trying to reinvent existing knot theory, just to make a journey of intellectual and mathematical discovery. The previous article described how I gently shocked the children by showing them that mathematics is not only about numbers, but about patterns in general, such as knots. Being eight-year-old children, they are very theatrical in expressing emotion, “What! Maths about knots?” I loved the wild face expressions and hand gestures. We spent a lot of our time making and drawing knots and concluded that drawing knots is quite hard. We needed a better notation.

A notation for knots: the “knotation”

One preliminary observation that I made, which resonated with the children, was that there are a lot of numbers out there but we only need 10 symbols to represent all of them: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. Of course, we could only use 0 and 1, or just 1, but let us not get distracted. It is about having a nice and convenient notation not just any notation. So, I drew a couple of knots on the whiteboard and I encouraged the children to try to identify the “smallest pieces the knot is made of”, where by a “piece” I meant this: We draw a small circle on the whiteboard and the “piece” is the bit of knot we see inside the circle. If our circles are small enough it is easy to see that the pieces inside the circles are quite similar. So here is an overhand knot with some parts of it circled (see figure 1).

We noticed that even if we ‘zoom in’ on a knot piece, as in the case of the red circles, the interesting knot piece ‘looks the same’ (see figure 2):

![Figure 2: A knot piece.](image)

We noticed that if a knot piece is too large, as in the case of the blue circle, it can be decomposed into smaller basic knot pieces. We also noticed that many knot pieces, which we select here and there, look quite similar, as in the case of the green circles.

I do not think there is a clear methodology one can use in inventing a good notation for a concept or set of concepts. It is a fun and creative endeavour. It is a design exercise in which many criteria must be balanced.

On the one hand we have expressiveness, which means that we can use the notation to represent a large range of concepts. But we also have elegance, which is more subjective but not entirely arbitrary. We want a small, but not too small, set of notations. We want it to be also concise. We want it to be pretty. So, we explored a while and in the end, with some careful guidance, we narrowed in on these constants (see figure 3):

![Figure 3: A “knotation”.](image)
The names C and X were suggested because of the shape of the knot piece they represent. L was short for "line". These shapes can be flipped over and result in 3 other constants (see figure 4):

![Figure 4: 3 further constants.](image1)

Here there was some disagreement between me and the children. They were happy with the flipping operation and they were happy with X* but they did not like C* because it looked more like a D, without the vertical bar. Some of them insisted that we call them D and X*. L* was a non-issue because it was just like L. I put my foot down and we settled on C, X, L and the * operator. This exercise also presented us with our first equation: L = L*. I did not insist on equations as they will become more important later. The final part of the hour was dedicated to developing notations for assembling knots out of these constants. There are two ways to compose knot pieces, horizontally and vertically. Here is how the overhand knot can be split into basic components (see figure 5):

![Figure 5: Splitting an overhand knot.](image2)

How do we put them together? The bits at the top can be written as XXXX. The bit below seemed like C-C*. And at the bottom we have L. Because the bit at the bottom is quite long many children wrote it as L-L or L-L-L but they quickly realised our second equation, L = L-L. This observation led to an unfortunate turn of events in which the children decided that L is like 0 because added with itself stays the same, and therefore changed it from L to 0. I decided that in the next session I would change it to the correct one even if I need to exercise my executive power.

Then we ran out of time without designing a good notation for vertical assembly. Because we did not have vertical assembly we could not really test our notation. The way C, C* and L are composed is not entirely clear because vertical and horizontal composition interact. But, we had made a start. I wanted the children to move away from how they are used to studying mathematics, as a stipulated set of rules in which answers are either correct or incorrect. I did not want them to simply execute the algorithm, but to invent the programming language in which it is written. Coming up with stuff that does not work is okay, as long as we notice it does not work and we improve on it.

What did we accomplish? Quite a lot. In one hour, a dozen eight-year-old children rediscovered some of the basic combinators used in monoidal categories, such as the braid, or such as duality, unit and co-unit as used in compact closed categories. We also rediscovered composition and its identity, and some of its equations. We used none of the established names and notations, but we discovered the concepts. We were on our way to inventing a categorical model of knots and braids. The next article will describe how we figured out the monoidal tensor and moved on to the really exciting bit, coherence equations.

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This article first appeared as a blog in 2015. The original blog is no longer available. The club was held at Water Mill Primary School in Birmingham.
Observation lessons began in the early 1960s and seemed to be a purely ATM phenomenon. They became a regular feature of my work in professional development.

I had two rules for the observers. The first, “Do not talk during the lesson”, seemed justifiable. The justification for the second rule was often less clear until the lesson began: “Please don’t interfere.” On the few occasions I forgot to issue this rule, someone always interfered!

What exactly was the purpose of an observation lesson?

Among other things, it was an opportunity to watch children working at some mathematics, unhampered by the responsibility for the lesson. However, it was all very well asking observers to watch the children, but it was inevitable that they also watched the teacher; and indeed they should be doing that too, because the lesson was the result of continual interaction between teacher and children. This interaction was something that was important to discuss. I shall return to this later.

It was sometimes necessary to point out the so-called artificiality of the situation, in which I conducted a one-off session, usually with a smaller group of children than normal, that I had not met before and probably would not see again. (Geoff Sillito, in a lesson at an early conference, dealt deftly with the question that often arose after a lesson, “What would you do next?” “Oh,” said Geoff, “I hadn’t planned to do anything next; I knew I was only going to see the children for one session!”)

Also, this already artificial lesson was surrounded by a group of observers. Furthermore, they were expecting something extraordinary, or why would we all be there together doing this? If it were to provide a stimulus to discussion, then it should provide something worth discussing. And it was here that the essential artificiality of the situation occurred in two different ways.

First, one could not always treat an observation lesson like a classroom lesson, where perhaps a large part of the time the children might be working quietly on their own or in pairs or groups, because then there was nothing for the observers to observe. This meant, then, that a lesson had to be conducted so that there was always something happening that could be observed.

The second, and more positive, aspect of artificiality was that one had the option of manipulating the situation, in a way one would perhaps not do in one’s own classroom, in order to demonstrate certain things for the observers.

For instance, I was working with a whole class of 7-year-olds in Belfast and they were making triangles on 9-pin geoboards and discussing them. This was going well as a demonstration of what ideas the children had about classifying triangles and the properties that they were noticing. Then in the course of conversation it transpired that one of the triangles, in a particular orientation, was, in the opinion of the class, not a triangle. We discussed the number of sides and they agreed that a triangle had three sides, and even that three-sided shapes were called triangles, but this did not make the triangle in question a triangle. It occurred to me to try to demonstrate to the observing teachers something about the impossibility of giving 7-year-olds definitions:

“Look,” I said, “Any shape which has three sides is always called a triangle. Whatever it looks like. So if a shape has three sides, what do we call it?”

“A triangle.”

“Is this a triangle?”

“No.”

“How many sides has it?”

“Three.”

“What do we call a shape with three sides?”

“A triangle.”

“So is this three-sided shape a triangle?”

“No!”

The children luckily responded in a way that was helpful to the point I wanted to make, but it is not always so easy, and in an observation lesson one
is always at the mercy of the children. Most of the time the children rose to the occasion, but I had some lessons when it was not all plain sailing. Once or twice the class were slow in coming up with anything. And on a couple of occasions one pupil came up with so many interesting ideas that they seemed worth pursuing, to the exclusion of the others. But I had a hard time justifying this to the observing teachers, some of whom wanted to preserve at least that natural element of normal classroom practice that required that the teacher made sure that all children participated.

The degree of participation became an interesting point of discussion on occasions, when we found that we could not actually agree about who had participated and who had not. “That girl took no part at all,” said one observer. “Ah,” said another, “but I was sitting opposite her and I could see that she was doing a lot.” Where you were sitting seemed to be crucial.

Teachers also began to recognise that children could participate in different ways. A verbal contribution is not the only possibility: children communicate by gestures, or by actions on the materials they are using. Judgement about participation thus became a matter of what was fact and what was conjecture. There were definite signs of participation, like speaking and demonstrating or gesticulating. There were other signs that enabled one only to make conjectures, like facial expressions, smiles, nods, frowns. One could choose to make an issue of some of these things during the lesson, by trying to draw out the reticent child, or containing the talkative one, or encouraging the child communicating with gestures to verbalise.

Maybe what was important was for observers to learn something about the business of observing and the limitations put on it by such factors as the geography of the room, or whether you had responsibility for the lesson, because it invariably transpired that I as the teacher knew less about what went on than any of the observers. All this could be salutary in getting teachers to consider how much they really knew about what went on in their own lessons.

These points about participation often arose early in the discussion which followed the observation. Even if nothing unbalanced occurred someone would comment, “The girls took part just as much as the boys.” It then became a problem for me to steer the discussion towards what the children actually said or did by way of mathematical activity.

This was never easy. There could be as much disagreement about what happened in this respect as there was about the level of participation. That is, there could be differences of opinion about what was said or done. Memory could play tricks, of course, and some teachers would support their claims by reading out the notes they made during the lesson. Sometimes there was conflict between the records of different observers. The whole process of ascertaining what happened was thus shown to be somewhat precarious, as a result of the simultaneous observing of people with different physical viewpoints, different expectations or different preconceptions about children. This was again an unnerving challenge to the cosy feeling that in one’s own classroom one knew more about what was happening, because there was no-one there to contradict; and the advantage of knowing the children could be lost in the preconceptions that arose from that very knowledge.

Teachers could thus begin to appreciate that there might be different accounts of what actually took place in a lesson and that the facts themselves, in terms of what children said or did, could be difficult to ascertain and must be handled with care. They could more readily accept that there would be different opinions about what conclusions could be drawn from the observed facts.

One of my never-ending problems, however, was to get people to realise what was fact and what was conjecture. “How do you know that happened?” I often asked. “Because I saw it happen,” was a different answer from “Because I saw something else happen and so I assumed that happened.” Once we had managed to sort out the difference between fact and conjecture we could then see that participation, for instance, was very often a matter for conjecture.

The conjectures, in spite of their greater scope for error, were the most important aspects of observation, because all forms of evaluation, assessment, testing and diagnosis relied on them. We could only know what ideas children had by forming conjectures as a result of what they said or did.

Often the ‘obvious’ was a platitudinous and glib attempt at explanation. I sat in on a session in which teachers were watching a video of the television programme, Twice five plus the wings of a bird, first shown on the BBC2 Horizon programme in April 1986. One sequence showed a young boy adding five and three, by adding four and four. When explaining this to the interviewer he demonstrated with his
fingers, but had trouble finding the right words. The comment of one of the teachers was, “The language gets in the way of the mathematics.” I queried this, and asked if it was not rather that the mathematics got in the way of the language. Maybe that was not quite right either. However, it prompted a deeper discussion about what got in the way of what; what skills the child in question did or did not have and how the business of communicating one’s ideas was to do with far more than spoken language. The interviewer, and the viewers, understood what the child meant, so communication had been effected, even though the verbalisation was difficult. So, my intervention prompted a discussion about the nature of communication, as well as attempting to sharpen up the clarity of the conversation.

What I tried to do was to get teachers to behave more mathematically, more analytically, in their conjecturing. And often what was necessary was to say that as a result of what we perceived, then various consequences were possible. We might wish to attach probabilities to these possibilities, but that was more subjective. It was all to a certain extent subjective anyway, but I was trying to minimise the subjectivity by tightening up the process of deduction.

This process of deduction was something that I also expected from the children. A result of this expectation was that a first teaching principle was never to tell the children anything that they could deduce for themselves.

A consequence of this was that one did not correct mistakes. Once, in the course of an investigation about transformations on a grid of equilateral triangles with a class in my own school, someone suggested that two of the angles (of 60 degrees) made a right angle. This was accepted without question by the rest of the class. I did not interfere. The discussion continued. Some 20 minutes later a contradiction arose. The class traced back their arguments and realised they had made a wrong assumption about the right angle.

The principle of not correcting errors was not solely due to general ideas about who has control of learning, though this was obviously important. It also has considerations peculiar to mathematics. It relates to the principle of children being able to deduce things for themselves, because, as the example of the right angle showed, it was possible to deduce errors from subsequent contradictions that arose. This, I believe, is a skill that children should be allowed to develop from when they first start to learn mathematics.

It is also desirable that children develop a critical approach to other people’s mathematics. So, it was a natural part of my teaching always to invite ideas from the class and then to ask for judgements about the ideas offered, with criticism and justification, argument and counter-argument. The children had to decide what was correct or not. My role, after presentation of the initial problem or situation, became that of chairperson rather than that of arbiter.

In the normal classroom setting it was probably difficult for me initially to avoid making judgements on the responses of the children. I had to develop techniques for being non-committal. One has only to read John Holt’s How children fail (Pitman, 1964) to see how easy it was unwittingly to betray one’s opinions with body language or voice inflection, by being selective, or by biasing one’s reactions. So early on I worked on the necessary techniques until I thought they had become second nature. But I had a small shock one day from my sixth form, most of whom I had been teaching for six years. I had asked a question, and I had written someone’s answer on the blackboard, and as usual I was waiting for them to decide whether it was correct or not. Bill said to Graham, just loudly enough for me to hear, “It must be right, because he’s got the chalk in his right hand and the blackboard rubber in his left!” I just laughed and never followed it up, and I like to think that Bill was pulling my leg, but sometimes I have wondered whether all my efforts were in vain, or could be undermined by the habits of the class’s other teachers, or even whether Bill had been reading John Holt!

What sometimes upset teachers, though, was the way an observation lesson could be left hanging in mid-air, so-to-speak. I once began a lesson with a group of ten-year-olds with the statement, “The answer is 10, what is the question?” They were eventually working on all possible ways of adding two numbers to obtain ten, and had justified their conclusions, when one boy said, “What about seven-and-a-half and two-and-a-half?” “Two-and-a-half is not one number, it’s two numbers,” said another; “two, and a half.” “No, it’s three numbers,” said a third, indicating the whole number, the numerator and the denominator. The argument continued. I concluded the lesson as I usually did merely by thanking them for coming, but they were still arguing fiercely as they walked out of the room. I suggested to the observing teachers that it was a nice way to end a lesson, and they agreed.

It was different with a group of primary inspectors, who
as part of their own in-service day on mathematics watched me work with a group of ten-year-olds with calculators. We worked on a series of problems. It was clear to me that the last one was going to be a little difficult for them, so I more or less said so, and suggested that as it was time for their lunch they go off and think about it later if they wanted to. This raised the anger of some of the inspectors, who seemed mainly to be concerned about the worry it would give the children because they had gone away without solving that last problem. I found it very difficult to persuade them that in the first place it was unlikely that they would remember anything about the problem once they got out of the door, but on the other hand that it might be nice if they went away sufficiently interested in it to want to carry on working on it. But I felt that the real worry on the part of the inspectors was the feeling that things had been left in mid-air and that an apparent concern for the children disguised their own discomfort which was caused by the lack of resolution.

I was engaged in helping with a course at another mathematics centre, and after one of my days, which included a lesson with children, I sat down with the course tutors to discuss it. One of them suggested I had “threatened” the teachers, and we discussed why. The reasons seemed to have something to do with the way I taught, not telling, not saying whether things were correct, not praising.

We discussed the difference between “threatening” and “challenging”, and the idea that which of the two it was depended perhaps on the confidence of the person on the receiving end. But when I asked what the purpose of in-service education was, one answered that it was to change teachers. I would not have put it quite like that myself, but I did suggest that if that were the aim then one could not achieve change without implying in some way that what the teachers were already doing needed to be changed, and hence the need at least to challenge.

The way I would put it, and indeed used to put it to teachers, was that the purpose of my standing up in front of them and teaching children was not that they went away and taught like me, but that they used what they saw in order to think about what they did themselves.

That is, of course, admitting that they inevitably were going to observe the teacher. But I still wanted them to concentrate on what the children do.

David Fielker was Director of Abbey Wood Mathematics Centre, London, 1967-89 and Editor of MT, 1972-83.

A longer version of this article was published in For the learning of mathematics 10(1), Canada, 1990.

NEW

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Cuisenaire rods and symbolic algebra

Dietmar Küchemann discusses the use of Cuisenaire rods to support students' developing understanding of symbolic algebra.

Figure 1a shows a time-line marked in equal intervals, to represent days and weeks. We can make this more concrete by placing ‘rods’ along the line (see figure 1b). And we can label the rods (see figure 1c).

Using these labels, we can construct equations of the type, $1W = 7D$, $3W = 21D$, and so on.

Here the rods have been labelled in much the same way as Cuisenaire rods are commonly labelled. This has an algebraic look about it, though it may not be immediately obvious where the generality lies. In one sense, the equations are all saying the same thing, that the time interval $W$ is 7 times the time interval $D$. So, we can reduce the equations to just one, of which the simplest is $W = 7D$.

This equation is general, in as much as it is saying that, whatever unit we choose to measure $D$ and $W$, the value of $W$ will always be 7 times the value of $D$. For example, if we think of $D$ as 24 hours, then $W$ is 168 hours. If $D$ is 0.5 weekends, $W$ is 3.5 weekends.

In a second sense, the set of equations is general in that there is a relation between the number of $W$s, call it $w$, and the number of $D$s, call it $d$, that holds for each equation. If we ask, “What is the relation between $w$ and $d$ in $wW = dD$?”, we get this (or an equivalent) equation: $d = 7w$.

You might be feeling a bit queasy at this stage. The relation $d = 7w$ is a general statement, with $d$ and $w$ acting as genuine variables, without having to bring units of time-measurement into the story. For example, it expresses the fact that when $w = 6$, say, then $d = 7 \times 6 = 42$. However, it seems to clash horribly with our earlier, more specific and more grounded statement $W = 7D$, and, indeed, with an everyday sentence like, “A week is 7 days”.

This confusion can be difficult to unpack, but it rests on the fact that the letters $D$ and $W$ are representing something very different from $d$ and $w$. We used $D$ and $W$ to represent quantities. Initially we used them to represent the length of two kinds of rod, where each $W$-rod is 7 times as long as a $D$-rod. In turn, we used the rods to represent time-intervals, where the interval $W$ is 7 times the interval $D$.

In contrast to this, $d$ and $w$ are pure numbers: $d$ represents the number of $D$-rods needed to match the number, $w$, of $W$-rods; or $d$ represents the number of time-intervals of a day’s length that make $w$ time-intervals of a week’s length.

Returning to our time-line and the rows of rods, instead of labelling the diagram as in figure 1c, we can label it as in figure 2a, or as in figure 2b, where we have dispensed with the rods.

Figure 2b, or figure 2a, is mathematically richer and more powerful than figure 1c, and shows the kinds of scale we use to represent multiplicative relations on the double number line, or functions on a mapping diagram or Cartesian graph.

**Letter as number, as quantity, or as object**

Sometimes the use of a letter to represent a numerical quality of an object can slip into using the letter to stand for the object itself, as in the use of “$a$ for apples, $b$ for bananas”, when manipulating algebraic expressions. Sometimes such a non-mathematical use of letters works, in that it does not throw up obvious or immediate contradictions. Indeed, it works in our time-line context,
where a sentence like $5W = 35D$ can be read as “5 weeks equal 35 days” without causing any obvious difficulty, even though mathematically it really means something like “5 units of time 1 week long is the same amount as 35 units of time 1 day long”. Or consider the example in figure 3, taken from W. W. Sawyer’s *Mathematician’s delight*. Sawyer begins with a story involving the cost, in old pence, of some buns and cakes. We are told that “two buns and a cake cost 4d” and that “three buns and two cakes cost 7d”. In figure 3 he shows how we can apply common-sense notions to this information and thus find the price of each bun and each cake, and how we can do the same by forming and solving simple simultaneous equations. Without coming to any harm, Sawyer moves seamlessly between correctly using the letters $b$ and $c$ to stand for the price of various objects, namely a bun and a cake, and wrongly matching them to the objects themselves. Now consider the task in figure 4. Please work through the task before reading on.

It is tempting to interpret $8c + 6t$ in figure 4 as 8 cabbages and 6 turnips. You may well have done so yourself initially. This interpretation leads to the conclusion that the total number of vegetables bought is 14 rather than $c + t$. When this task was given to some 13-14-year-old students (see Küchemann, 1982) only about 1% explicitly stated that the given expression represents the cost of $c$ cabbages and $t$ turnips. Only 4% wrote $c + t$ for the total number of vegetables. Results on an equivalent task in 2008/9 were similar.

Another issue in using Cuisenaire rods and the Cuisenaire letter notation in the teaching of algebra stems from the fact that the rods are fixed. One can get round this, to some extent, by not numbering the rods and imagining that they have been scaled. However, this leaves the relative values of the rods unchanged, which is not always what one wants. What one cannot do is simply stretch an individual rod and thereby use it to model the notion of variable. Consider the equations in figure 5. In an informal study that I undertook (see Küchemann, 1983), it turned out that lower secondary school students were far less successful in solving equation A than equation B, or even equation C. I leave it to the reader to consider why.

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**Letter as variable**

Thus, in using Cuisenaire notation, we need to be aware that we are, primarily, using the letters in one particular way, to represent quantities rather than pure numbers. Further, though this usage is perfectly legitimate, we need to be aware that it can easily morph into using the letters as representing the objects themselves. Though this can work in some mathematical situations it can be entirely ill-fitting in others.

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**Figure 3: Mathematician’s delight pages 79-80.**

**Figure 2a: Relabelled time-line with rods.**

**Figure 2b: Relabelled time-line without rods.**

**Figure 4: Cabbages and turnips task from CSMS Algebra test.**

**Figure 5: Examples of equations to solve.**
Cuisenaire rods and symbolic algebra

A model where rods can stretch, provides a way in to equation A. We can model the left-hand side and the right-hand side as in figure 6, where the shown value of \( e \) is 2. We can then ask, “For what value of \( e \) are the two rows of rods the same length?”

Not only does this offer a way of making sense of the equation, it engages students with the notion of variable, as opposed to just thinking of a letter as representing a specific unknown. It can also help students develop a feel for how the various elements in an expression (e.g., the 3 and the 5 in \( 3e + 5 \)) ‘work’. I have been using this kind of dynamic approach in lessons developed for the ICCAMS project directed by Jeremy Hodgen. A movie of dynamic rods can be found at https://www.youtube.com/watch?v=ZbzXGyTpM28.

Using Cuisenaire rods

How might these issues impinge on the actual use of Cuisenaire rods? Some interesting tasks are shown in the centre pages of MT256. One task involves the observation that a yellow rod is the same length as a green rod joined to a red rod (see figure 7).

Figure 7: Task involving rods.

The text suggests that this can help students see and record relations such as \( y = g + r \), \( y – r = g \) (and others such as \( y = r + g, r + g = y \)). The aim, of course, is that students will begin to see the relations generically. In other words, given any relation of the form \( p = q + r \), where the letters now represent genuine variables, students will see that relations such as \( p – r = q \) also apply. This raises the interesting question, do we help students achieve this aim by using semi-abstract statements like \( y = g + r \), despite the limitation that the letters are not proper variables, or should we simply use the numerical relations embodied by the rods, in this case \( 5 = 3 + 2, 5 – 2 = 3 \), as the springboard for the generalisations? I will not offer a definitive answer to this but the next activity may help clarify this issue. Figure 8 shows an initial response to another task featured in the MT 256 extract, which involves the use of rods to construct equivalent squares. There are lots of ways of seeing the squares generically. One possible structuring is shown in figure 9. Algebraically, it can be represented as \( n^2 = (n – 2)(n + 2) + 2^2 \).

Figure 8: An equivalent squares task.

The symbolisation shown in Figure 8, that is \( 6D = 4W + 8P \), is interesting here. Consider the expression \( 6D \). This accurately describes the dark green square, as consisting of 6 lots of dark green rods of length \( D \). However, it does not fully describe its structure, namely that we do indeed have a square and that it consists of \( D \) lots of rods of length \( D \). So \( D \times D \), or \( D^2 \), would be a better description, though it involves interpreting \( D \) in two ways simultaneously, as a pure number and as a quantity, length. This can be challenging, but it is something that students need to grapple with as they develop their understanding of multiplication.

A second challenge stems from the phenomenon noted before that the letters in the Cuisenaire notation are not variables. They represent specific lengths. So, for example, if we are trying to express the structuring of the square shown in figure 9, then the term \( 8P \) in the above equation does not make the structure explicit. It merely tells us that the square made from dark green rods can be re-formed as a square made of 8 pink rods of length \( P \), plus 4 other bits. The key is not that the rods are pink or that there are 8 of them, but that their length is 2 units less than the green rods, so \( D – 2 \), and that there are 2 more than the number of green rods, so \( D + 2 \). Notice how \( D \) again has two meanings here,
a quantity and a pure number. By working in this way, that is, by writing all those things in terms of \( D \) that vary as \( D \) varies, it is possible to arrive at \( D^2 = (D + 2)(D - 2) + 2^2 \), which has the same form as the relation in the variable \( n \), above. But, to achieve this, we have to cope with a further ambiguity, of treating \( D \) as both fixed and changing. So, \( D \) is now acting as a quasi-variable, but in a somewhat indirect way. We can think of the letter \( D \) as standing for any other letter used to denote any, actual or imagined, Cuisenaire rod and in turn for any number that can be used to denote the length of such rods.

We might relish the opportunity to confront students with this ambiguity and complexity when using Cuisenaire notation. On the other hand, if our prime aim is to look for structure and to express this algebraically, then it might be simpler to use plain numbers as quasi-variables, by constructing open number-sentences in this kind of way:

- Start with the 6 by 6 square, say: \( 6 \times 6 = 6 \times 4 + 2 \times 4 + 2 \times 2 = (6 + 2)(6 - 2) + 2^2 \).
- Check by visualising a 20 by 20 square, say (as in figure 10): \( 20 \times 20 = 2 \times 18 + 20 \times 18 + 2 \times 2 = (20 + 2)(20 - 2) + 2^2 \).
- Express in terms of \( n \): \( n^2 = (n + 2)(n - 2) + 2^2 \).

**Conclusion**

Cuisenaire rods provide a rich resource for developing students’ understanding of number. However, if our aim is to develop students’ understanding of symbolic algebra, then I think care needs to be taken when using the standard letter notation associated with the rods. In this notation, a letter is primarily used to denote a quantity, so students need to become aware that letters are commonly used to represent pure numbers in school algebra. Also, where a letter denotes such a quality of an object, for example, its length, mass or price, it is easy to let the meaning slip, so that the letter is used to denote the object itself. Such usage may not cause any perceptible disruption in contexts where Cuisenaire rods are commonly used, but students need to become aware that there are other contexts where such an interpretation does not fit. Further, the fixed nature of Cuisenaire rods and especially the fact that we cannot stretch individual rods, means that Cuisenaire notation is not always well suited to expressing generality. It might sometimes be simpler to assign numbers to the rods and use these to construct open number sentences rather than use the rods’ letter-names as quasi-variables.

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**References**


The curious convention for classifying triangles

Nat Banting triggers classroom debate by exposing a geometric inconsistency with the conventional definition of the scalene triangle.

Several times throughout my career, my class and I have stumbled our way into the contentious debate of whether a square can be considered a rectangle, a rectangle a square, or a square is simply a square and a rectangle is simply a rectangle. My interest in this debate has not waned over the years, however, my stance on its resolution has changed considerably due to information brought to my attention through, arguably, the most unimpressive of shapes: the scalene triangle. Here, I discuss how a classroom debate about the geometric properties of triangles influenced the debate surrounding four-sided polygons and deepened my students’ understanding of mathematical definition and classification. In order to mimic the experience of my students, I encourage you to take a moment to establish your stance on the relationship between squares and rectangles before reading on.

**A conventional solution**

For years, I invigilated classroom debate with a false ear. That is, I listened to the dialogue with the explicit intent of eventually having the students arrive at what I felt was the inevitable conclusion. To me, there really was no room for debate, no tension between a reasonable alternative and the conventional wisdom. A square was unquestionably a rectangle because of the hierarchy of necessary conditions that establishes more specific quadrilaterals as subsets of more general ones as additional conditions are imposed. For example, a quadrilateral must have specific properties in order to be considered a rectangle. The opposite sides must be parallel and it must contain a right angle. A square adds a single extra requirement to those of the rectangle: it requires opposite sides to be parallel, a right angle, and a pair of congruent adjacent sides. In other words, every square satisfies the requirements of both rectangles and squares. Going beyond the necessary conditions for being classified as a rectangle does not jeopardise either classification. Any resistance I provided to my students was artificial in an effort to trigger student conversations, and any vibrancy of the debate was often sponsored by students’ differing levels of geometric thinking. I never believed that there was an accepted alternative argument until I recognised that a beautifully contentious, and familiar, example was sitting in plain sight the entire time: the scalene triangle.

I ask you, just as I would ask my students, to consider the common definitions of scalene, isosceles, and equilateral triangles in hopes that you find them both familiar and bothersome:

- A scalene triangle is a three-sided polygon in which all three sides have different lengths.
- An isosceles triangle is a three-sided polygon in which two sides have the same length.
- An equilateral triangle is a three-sided polygon in which all sides have the same length.

The set of equilateral triangles is a subset of isosceles triangles according to the definition above, because having all sides of equal length satisfies the requirement of having two sides of equal length. This relationship is isomorphic to the one discussed earlier between the square and the rectangle. However, the scalene triangle is excluded from the hierarchical relationship shared between the rectangle and the square, because the requirement that all sides be of different lengths is not satisfied by an isosceles triangle. This is a curious way to define the scalene triangle because isosceles and equilateral triangles, and all quadrilaterals for that matter, are defined based on the properties they have, but the scalene triangle is defined based on the properties it must not have. For whatever reason, the definition of scalene triangles is inconsistent with all other accepted definitions for triangles and quadrilaterals.

The inconsistency of the scalene triangle legitimises an alternative voice in the quadrilateral debate for two reasons. First, the dissenting argument is based on mathematical structure, not a developmental level. The argument is now between two competing systems of definition, both of which are correct. Second, this new exclusive definition is widely accepted by mathematical convention in the case of triangles. It is not just fabricated by a teacher to stimulate debate. It is ingrained in mathematical convention, a juxtaposed opinion hiding in plain sight. It is oddly familiar yet widely contentious, the perfect recipe for classroom activity.

**A curious solution**

Investigating this difference provides space to wonder why this is the case and what it might look like if we used a more consistent system of classification.
In the spirit of resolving the tension, I present two alternative options for defining triangles.

- A scalene triangle is a three-sided polygon with at least one side of equal length.
- An isosceles triangle is a three-sided polygon with at least two sides of equal length.
- An equilateral triangle is a three-sided polygon with at least three sides of equal length.

This set of definitions is intended to mimic the accepted system for quadrilaterals. All that has changed is that now, the scalene triangle is also characterised by the properties it contains, rather than by the properties it is forbidden to have. Students often contest that the scalene definition is trivial. Of course every side of any polygon is the same length as itself. I contend that this is not an issue, because it still satisfies the basic requirement of being a triangle. It has three sides. I am also quick to point out that they have no problem with the analogous case in quadrilaterals.

I also present a second set of definitions that extends the exclusionary style of the conventional definition of scalene triangles to isosceles and equilateral triangles.

- A scalene triangle is a three-sided polygon with exactly zero pairs of sides of equal length.
- An isosceles triangle is a three-sided polygon with exactly one pair of sides of equal length.
- An equilateral triangle is a three-sided polygon with exactly three pairs of sides of equal length.

All three of these definitions now define a given triangle by what it must not have. Take the new definition of an isosceles triangle. Although it feels like it is describing what it must contain, one pair of congruent sides, by requiring that it contain exactly one pair, it also ascribes what is must not contain, zero or three pairs of equal sides. This style forces each type of triangle to be classified in isolation, rather than as a member of a hierarchy and extends the same exclusionary treatment of the scalene triangle to all three types of triangles.

Given this new information, I encourage my students to debate the merits of both new systems and apply their lines of argument back to the quadrilaterals. Introducing the curious case of the scalene triangle focuses the debate on the differences between the two possibilities and forces students to determine, justify, and employ their convention of choice. When we question the definitions, students begin to understand that the definition of a shape is far from standardised. This interrogation, rather than blind acceptance, leads learners to active engagement in abstract geometrical reasoning. Both sets of new definitions seem to appeal to conventional sensibilities in some manner, but contest them in others. This dissonance is extended to the classification of quadrilaterals, where they must debate the appropriate definitions and subsequent classifications of squares, rhombi, rectangles, parallelograms, trapezoids and the like. What follows is a description of the arguments from two students, Foster and Elia, when asked to respond to the following prompt after our classroom debate about the scalene triangle:

Provide a convincing argument for one of the two statements: A parallelogram is a trapezoid or a parallelogram is not a trapezoid.

**Foster: An evolutionary argument**

Foster wrote in favour of the conventional hierarchy of shapes and was strongly against the unnecessary segregation required by the second set of proposed definitions. He entitled his response *The evolution of the quadrilateral* and used evolutionary language in supporting his claim that a parallelogram was indeed a trapezoid. According to Foster, if a trapezoid requires one set of parallel lines, a parallelogram with two sets of parallel sides should be considered a trapezoid, albeit a more evolved trapezoid. He began with a drawing of a generic quadrilateral (see figure 1), indicating that “there is nothing special about this shape.” Directly below that, he wrote, “the offspring gains one set of parallel lines” next to a drawing of a trapezoid. The quadrilateral and trapezoid were then connected by an arrow depicting the first step in the evolutionary pathway, signalling an understanding of logical ordering.

![Figure 1: Foster's evolutionary argument.](image-url)
The curious convention for classifying triangles

metaphor of biological progress. To create an isosceles trapezoid, “One set of congruent lines develop,” and then the parallelogram is born when “the congruent lines shift to create two sets of parallel lines.” For Foster, a shape is a product of properties and the evolution of properties meant the addition of possible classifications. Gaining new properties did not exclude a shape from retaining the classification as a former, more primitive concept. By drawing out the evolution of the quadrilateral by adding properties, he showed understanding of necessary properties, geometric definitions and logical ordering. When I asked him if his metaphor extended to the triangles, he was consistent in his thinking and thought that the definition of the scalene triangle should be altered to reflect his evolutionary design.

Elia: Two does not equal one

Elia initially stated that she thought an equilateral triangle was a special isosceles triangle, but also claimed that a triangle with three equal sides did not have two equal sides, it had three. Therefore, at the onset of the class discussion, she actually held two opposing views simultaneously. This suggests that she had previously encountered the definitions of triangles as arbitrary facts to be memorised. Classroom debate provided Elia the opportunity to consider a suitable, and consistent, classification based on the properties of the shapes and she ultimately reversed her support for the evolutionary classification presented by Foster based on a conviction that “one and two are very different things.” Elia justified her claim that a parallelogram is not a trapezoid on the basis of this exclusivity of number.

Figure 2: Elia’s exclusive levels.

According to Elia, unique shapes are created from a specific number of properties, and her response classified shapes into tiers based on the number of sets of parallel sides they contained (see figure 2). Adding or subtracting a pair of parallel lines created a new shape at a new level. This levelling was exclusive in the sense that it contained some sort of geometric amnesia where the old classification was rescinded in favour of a new one. In her words, “if you have two sets of parallel lines, you no longer have one” and therefore “the parallelogram and rectangle both have two sets of parallel lines, but it still doesn’t make it a trapezoid because that’s not how logic works.” She used this system of levels to work with parallelograms and trapezoids in both directions. A trapezoid is not a parallelogram because “a trapezoid only has one set of parallel lines and in order to be a parallelogram, it has to have two.” She then explained that, “parallelograms can’t be trapezoids because it has [sic] more than one set of parallel lines.” In general, parallelograms and trapezoids are distinct because they exist in different levels. For Elia, the case of the scalene triangle was potent enough to apply not only to the entire set of triangles but also the entire set of quadrilaterals and she built her classification system to consistently apply this style of definition.

Concluding remarks

The case of the scalene triangle serves to remind me that there are interesting mathematical conversations to be had in the most innocuous of places. The responses of Foster and Elia represent the two camps in the debate, but it is crucial to reiterate that the purpose of the debate is not to determine if one system is more correct than the other. Rather, it is to use the inconsistency of the scalene triangle to encourage the understanding that shapes are composed of properties and an analysis of how those properties interact to form relationships between geometric shapes.

It is worth noting that evidence of these geometric understandings persisted after the written responses to the task were collected. Weeks later, I attempted to limit student groups to the use of a single marker during a group problem solving activity. One group cheekily insisted that they were going to still take three, because, “having three markers is having one marker”, evolutionarily speaking. On another occasion, when an assignment dictated that students place seven features into a scale drawing, a student facetiously asked me to clarify whether I meant exactly seven features or at least seven features. The assimilation of this thinking into the everyday activities of the classroom provides further evidence of the deep mathematical activity sponsored by the closer analysis of the scalene triangle. In this way, the scalene triangle, which might be considered the most unimpressive shape of them all, has had a significant impact on my students, my own, and, hopefully, your geometric thinking.

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#beingmathematical

This article captures a discussion between Danny Brown, Mike Ollerton and Mary Pardoe arising from #beingmathematical.

#beingmathematical is a bi-weekly twitter chat hosted by @ATMMathematics during which participants are invited to work on a task selected from an ATM publication and then discuss what they did and the possible implications for their pedagogy.

Mary:

Back in 1985-6 I was reflecting on an article in my copy of MT110, Knowledge v experience by Caleb Gattegno, and in particular on these words: "In their [students'] experience, not their knowledge, are the insights which mobilise the proper know-hows which permit them to try this or that." (1985, p. 36)

How could I use this understanding in my teaching? At about the same time I bought Points of departure 1 (1980, hereafter referred to as PoD1) and started to select starting points from it for my lessons. I have continued to use many of these “points of departure” from all four books with many pupils and teachers, with the aim of providing experiences from which they might derive insights to, “mobilise the proper know-hows which permit them to try this or that.” However, I had never used Discs (see figure 1) from PoD1 because I thought it was too hard to get into. Taking part in a #beingmathematical twitter chat recently showed me how wrong I have been all this time.

Mike:

I am interested in your use of ideas from the PoD publications because this mirrors my relationship with them and how the tasks therein provided me with a wealth of starting points. Occasionally, I would try an idea out with a class without having first worked on it myself. I guess I intuitively felt certain tasks would have legs, mathematically speaking. In all, I have used ninety or so of the ideas many times both in my own classrooms and during CPD events. Indeed, I frequently send out a list of the 90+ ideas I have used, showing the youngest age of students I have used them with.

When I moved full time into initial teacher education, I used the Discs task in a problem-solving module on a 2-year PGCE Mathematics course; Discs was one of a choice of problems which students chose from and developed accordingly. In the context of #beingmathematical, I was aware, therefore, of the depths to which Discs could be taken and already had an idea of the relationship between the starting numbers and the required totals. However, all this happened a dozen and more years ago, so I felt to be starting again from scratch.

Mary:

I began by gathering a list and labeling the unknown values as a, b and c as follows (see figure 2):

![Figure 1: Discs, Points of Departure 1.](image)

I began by gathering a list and labeling the unknown values as a, b and c as follows (see figure 2):

![Figure 2: An initial labeling of the discs.](image)
From this starting point, I produced the following eight equations:

\[\begin{align*}
6+b+8 &= 23; \\
6+b+c &= 22; \\
6+7+8 &= 21; \\
a+b+c &= 18; \\
a+b+c &= 17; \\
a+7+c &= 16 \\
\text{and finally } a+7+c &= 15
\end{align*}\]

Knowing \(6+b+8 = 23\), then \(b = 9\).

From \(6 + b + c = 22\), then \(c = 7\); from \(a + b + c = 17\), \(a = 1\).

My disc set now looks like this (see figure 3):

![Figure 3: Relabeling the discs.](image)

Seeing the differences between fronts and backs were 5, 2 and 1, I kind of got mathematically excited when I explored what happened when the same differences were applied to each of the three starting numbers of 6, 7 and 8 and how each time the same solution set of 15, 16, 17, 18, 20, 21, 22, 23 emerged.

Thus, using this labelling (see figure 4):

![Figure 4: Final labeling.](image)

the minimum total, 15, is gained from 5+7+3 and the maximum total, 23, is gained from 6+9+8.

Mary:

During the #beingmathematical discussion I was able to find a solution. That is, by systematically listing possible outcomes, reflecting on what that showed me and as a result trying likely values, I soon came upon values for the numbers on the backs of the discs that gave the required totals; the solution set listed above. However, I had an uncomfortable feeling that I had just been lucky to chance upon a solution. I had no idea whether or not there were any other solutions, and why? Prompted by Mike’s conjectures I tried to convince myself that with three discs the sum of the differences must be equal to the difference between the largest and smallest possible totals when they landed. I was delighted when I eventually reasoned to a proof of this. It was only then that I really began to enjoy working on this task.

A crucial question for me is how to provide such opportunities for learners to feel that kind of satisfaction in doing mathematics. I am thinking about how to teach so learners become habitually on the look-out for generalisations about the truth, about which they might be able to convince themselves and others. This requires me to be aware of occasions when it is important to resist the temptation to do the thinking for the learner.

My reasoning in this case involved an important shift in how I thought about the discs. I abandoned the idea of front and back, which I had relied on in my systematic listing of possible outcomes and thought instead of two-sided discs with no side being more important than the other. I could then think about pairs of possible outcomes, numbers showing when the discs landed, that could not occur together. I thought of the numbers on one disc as \(a\) and \(a+p\), with \(p\) being the difference. There was no need to regard one of these as being on the front. The
smallest possible total when the discs landed is \( a + b + c \) and the largest possible total is \((a + p) + (b + q) + (c + r) = (a + b + c) + (p + q + r)\). The difference between the largest and possible totals is the sum of the differences between the two numbers on each disc. I got unstuck only when I stopped thinking about the numbers on the fronts, 6, 7 and 8, as dominating so that differences might be negative.

**Mike:**

I too checked out my last conjecture about the differences being 1, 2 and 4 and I gained the totals from 18 to 25 using 6/10, 7/5 and 8/7 and these have differences between fronts and backs of 4, 2 and 1 and my list of calculations now reads (see table 1):

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>( \Sigma )</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>5</td>
<td>7</td>
<td>18</td>
</tr>
<tr>
<td>6</td>
<td>5</td>
<td>8</td>
<td>19</td>
</tr>
<tr>
<td>6</td>
<td>7</td>
<td>7</td>
<td>20</td>
</tr>
<tr>
<td>6</td>
<td>7</td>
<td>8</td>
<td>21</td>
</tr>
<tr>
<td>10</td>
<td>5</td>
<td>7</td>
<td>22</td>
</tr>
<tr>
<td>10</td>
<td>5</td>
<td>8</td>
<td>23</td>
</tr>
<tr>
<td>10</td>
<td>7</td>
<td>7</td>
<td>24</td>
</tr>
<tr>
<td>10</td>
<td>7</td>
<td>8</td>
<td>25</td>
</tr>
</tbody>
</table>

Table 1: A table of differences.

This is ‘so’ binary! Column A having four 6s and four 10s; column B having two 5s then two 7s which repeats and column C having alternating 7s and 8s.

This now sits comfortably with your idea of \((a + p) + (b + q) + (c + r) = (a + b + c) + (p + q + r)\) and if I make \( p = 1 \), \( q = 2 \) and \( r = 4 \) the discs become \( \frac{a}{a+1} \), \( \frac{b}{b+2} \), \( \frac{c}{c+4} \) and the possible totals, in size order become:

\[
a+b+c; \quad (a+1)+b+c; \quad a+(b+2)+c; \quad (a+1)+(b+2)+c; \quad a+b+(c+4); \quad (a+1)+b+(c+4); \quad a+(b+2)+(c+4) \quad \text{and} \quad (a+1)+(b+2)+(c+4)\]

all of which simplifies to:

\[
a+b+c; \quad a+b+c+1; \quad a+b+c+2; \quad a+b+c+3; \quad a+b+c+4; \quad a+b+c+5; \quad a+b+c+6 \quad \text{and} \quad a+b+c+7.
\]

**Danny:**

We have a few different representations now: the “1,2,5 differences” description, Mary’s algebraic one and Mike’s table. To these I will add yet another representation, a tree-diagram which I imagined in my head (see figure 5):

**Mary:**

Danny, your attempt to construct a mental image of a tree-diagram to represent all the possible outcomes helped me to see why Mike was able to “move the differences about” with respect to the three given values on the discs without it affecting the possible totals. Generally, if I draw six tree diagrams showing all the possible pairings of the differences, \( p \), \( q \) and \( r \), with the given values, \( a \), \( b \), and \( c \), I see that, and how the set of eight totals is the same in each of the six cases. Here are the first two tree-diagrams that I drew, in which I interchanged \( q \) and \( r \) (see figure 5):

[Figure 5: Danny’s tree diagram]

[Figure 6: Mary’s tree diagram]
Danny:

Even with this representation, things remained foggy for me for a while. I started by thinking about the missing number, 19, but shifted to thinking about how to make the highest total, 23. I was stuck in the rut of thinking 8 had to be the highest number I could have on any disc, but gradually I realised that I must use a number higher than 8. Writing a 9 on the back of the 7-disc gave two possible totals, 21 and now 23 (see figure 7):

<table>
<thead>
<tr>
<th>Front</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>back</td>
<td></td>
<td></td>
<td>9</td>
</tr>
</tbody>
</table>

Figure 7: Danny’s first solution.

Almost instantly I became aware that I could shift these totals down by 1 to give two more totals (20 and 22) by writing a 5 on the back of the 6-disc (see figure 8):

<table>
<thead>
<tr>
<th>Front</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>back</td>
<td>5</td>
<td>9</td>
<td></td>
</tr>
</tbody>
</table>

Figure 8: Danny’s second solution.

I now had the four bigger totals, 20 to 23. I could shift these totals down by 5 to give the other four totals (15 to 18) by writing a 3 on the back of the 8-disc (see figure 9):

<table>
<thead>
<tr>
<th>Front</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>back</td>
<td>5</td>
<td>9</td>
<td>3</td>
</tr>
</tbody>
</table>

Figure 9: Danny’s final solution.

The problem started to be solved when I managed to jump out of the rut I was in, realising not only that I could, but that I must use a number higher than 8. I knew almost as soon as I imagined a 9 on the back of the 7-disc that it would lead to a solution. The problem seemed to unfold before me: how these two totals would become four and then eight, and how I could shift these totals up and down. I suspect the structure afforded by the image of the tree diagram allowed this to happen. Perhaps this unfolding-through-action is what Gattegno means by it being experience, not knowledge, through which insight and know-hows come about. I’m reminded once again of the phrase “laying down a path in walking” mentioned in my diary (p. 7)

Mary:

During the #beingmathematical discussion someone observed that, “Part of the mathematical excitement comes from finding a process for generating solutions rather than the particular solution itself,” which for me was significant. It reminded me of one reason why it is important that learners encounter tasks, such as Discs, that offer opportunities to search for, and devise for themselves, processes that provide certainty about results; about what they are doing. I felt the same excitement when I saw why Mike’s observation about particular differences was inevitable. Perceptions of certainty, where before there was uncertainty, that are achieved through one’s own thinking are a rewarding source of pleasure in doing mathematics: they are often sudden and usually preceded by mental struggle. Learners are owed opportunities to experience them.

In the same article, Gattegno writes, “The aim of teaching is to ease this acquisition of experience, which manifests itself in know-hows when applied to specific cases.” (1985, p.36) By working on mathematical tasks ourselves, sharing and reflecting on our experiences, we are likely to see more clearly how we can “ease this acquisition of experience” for learners including how to expand the range of tasks we offer learners. Doing mathematics, together from particular mathematical starting-points, helps us think deeply about what doing maths means in the company of others with whom we interact; how this may support our learning. So, we become more able to provide learners with experiences in which they can develop their natural reasoning powers; applying them to clarify situations and to solve problems.

Mike Ollerton is semi-retired but continues to be passionate about learning and teaching mathematics.

Danny Brown co-hosts #beingmathematical every second Thursday at 8pm: follow @ATMMathematics for more details.

Mary Pardoe has worked in mathematics education for more than forty years and continues to support mathematics teachers through her work as a NCETM Professional Development Accredited Lead.

References


From the archive: What matters most


It feels timely to share the views of the first President of the ATM. His challenge to teachers of mathematics 60 years ago feels as relevant as ever.

The main tasks confronting teachers of mathematics, and more especially, those in Great Britain, will be the subject of this, the first Presidential Address of the Association. Even if we did nothing to meet the challenges of the present and the future, no-one would blame us, but I shall assume here that quite a number of mathematics teachers, at all levels of instruction, have uneasy consciences and are prepared to examine their work critically.

Those who have decided to give their energy and their time to the meeting of these challenges, already know that they are engaged in a creative work of great importance. Teaching can be transformed, as can any other field of experience, into an area in which the creative qualities of the mind find their place and use. When this happens teachers feel that the reality they are meeting is larger than themselves, is inspiring and rewarding. It is extraordinary to find that so few teachers know that the situation in which they live and operate is endowed with such qualities.

We are in need of as many inspired teachers as there are classes. To these classes children are sent without their consent and are submitted to whatever we are as persons and what we can do with them. Only if we really care for them, and know what we are doing to them, do we justify our relation to the future and to the education of the next generation.

At present, as a profession, we are dismally ill-equipped. The Training Colleges and Departments of Education are doing what they can, but that is reduced to little because the necessary knowledge for meeting the challenges is either not yet available or is considered unimportant.

It is in order to answer some of the questions above, and to define some others that need consideration from the best among us, that I choose this theme for my address.

We must stop believing that the bright children are those who are verbally inclined and who manage to get into Public Schools (if they have the means) or into grammar schools. Brightness is a complex quality, and one dimension of it is emotional. If children are given what corresponds to their make-up, they become bright. If they are denied it, they give up the struggle to make sense of so many words. So that we have a share (and I think it is an important one) in the formation of the legions of the less bright pupils because of our preconceived ideas of how we should carry out our job of teaching. These need re-examination at once.

ATM has tried to concern itself with ways of doing things, with reaching pupils who usually shy away from words, and has questioned the entirely verbal approach to mathematics. In contrast to that attitude, as far as I can tell from my experience, the great majority of teachers are still doing in the classroom what was done centuries ago, unquestioningly following the syllabus and the suggestions of someone else. We must stop thinking that there exists a person hidden somewhere who knows exactly what each one of us has to do in his (sic) class at such-and-such a moment. The responsibility for teaching would seem obviously vested in the teacher who is at that moment there. But, in fact, it does not work out in that way. There is the
sylabus to cover, the many pages of the textbook to turn, the demands of several shadowy persons and institutions to satisfy before we consider the concrete situation of our class. The failures in the shadowy regions we interpret as our pupils’ or our own when, clearly, all that is at fault is an attitude: our loyalty. This we owe, as teachers only to truth and to the future. Big words! - but without what they cover we are nothing; or worse, we are frauds.

We must learn to take our share, fully, if possible, in the educational set-up. That is very important. We can satisfy inspectors, parents and examiners more surely by being part of reality, than by allowing all sorts of illusions to come in. (One of them can be singled out: the equating of ‘covering the sylabus’ and ‘our pupils have mastered its content’).

Only when we know better how to teach can we expect to lead our pupils towards mastery; we shall only know that when we know how our pupils learn. It is there that our main task is to be found. We have to study learning as a matter of course and all the time: not only in books, but where it takes place: in front of us, in the class room. Then we shall drop that other illusion of believing that each question has an answer. It may have; but, more likely, if it is a hard question it will require study and sleepless nights, and will only lead to one of many possible approximate solutions. Life questions are not as carefully schematic as the ones we meet in the textbooks particularly of mathematics. Teaching questions are complex and we have to learn to think in a complex way about complex questions. The other way is illusion again.

In our reality, people are involved. People are moody and unpredictable, demanding that we meet them without preconceived ideas. They want to be interested and challenged, and they dislike being taken for ‘things’, regimented, and their individuality forgotten. (Of course, some seem to like it. I call them anti-social because they have agreed to conform to what has emptied them of all their will to be themselves). Because we are dealing with people, teaching is a great art and we all should see it as such. It is a difficult art that requires us to work hard at it to be good at it. This is just the opposite of what teachers believe when they begin-and often even when they retire, for they, too, can be soulless as a result of ‘good’ behaviour for so many years.

Teachers must resolve to accept that only they, in their classes, at each instant, will find what to do that is right in the situation. No book, no lectures, can tell them, once and for all, what to do in all cases. If we are honest, we must admit that our ideas about teaching are not as articulate and full as they should be. We have clichés and slogans, and use them freely.

We are frequently resistant to new ideas and new methods. If someone comes with a proposal, and if it has not got final shape (a sylabus with suggestions, or even textbooks, examination questions, etc.), we do not want to waste time looking at it. In this way we so often lose a very good opportunity of improving ourselves and of giving better service. To close our eyes to reality because it appears in unfamiliar garments is very general. But can teachers afford that, since they do not work ultimately for themselves, but for children and the future? If teachers became sensitive to the uniqueness of their function and of its importance, they would look for what would help them to improve their work, and they would accept responsibility for using it in their own way.

Techniques are nothing if that background of sensitivity is not there; but they are everything with it. For then one can reach that level of awareness where one is able at once to translate into action what is known to be for the good of the students for them as individuals and for their future life.

The techniques I think of are those I know personally; my readers will think of others. But we must all be agreed that what matters most is that we reach a level of confidence based on knowledge, so that we can stand up for what we think and do, not because we have faith, but because it is true.

That is possible if we deliberately and with all the necessary energy pursue the knowledge needed in psychology, in mathematics, in methodology, and in experimental teaching. Not all will pursue everything; not all will need to. As an Association we can pool our experience and share it. That was the reason for its being formed.

In psychology, we are primarily interested in knowing what is the process by which mathematical experience is acquired; what makes it mathematical and nothing else; what are the various ways by which the mind creates those dynamic mental structures that become a power when one possesses them. We also want to know what is creating obstacles, which factors can be dispensed with, and which are indispensable (e.g. is sight necessary?). We want to know, too, if we can substitute one experience for another, and whether special techniques are needed to make the substitution efficient. (How far are words part of the mathematicalising process? Is
there a mathematical thought that exists outside all
the uses of senses, language, signs? Can we attain
it? Do we need to know of it to reach the handicapped
(sic) children properly? What problems do we meet
in these people whose universe of experience differs
from ours?) We want to know how long is needed for
such-and-such a mind to reach mastery of this or
that field of mathematical experience, and whether
personality, language, and the current environmental
modes of thought have parts to play.

These are some of our psychological problems, and
we can all help each other by finding parts of the an-
swers and by commissioning the specialists whose
additional help is required.

In mathematics, our needs as teachers are not
identical to those of scientists and technicians. What
we need most is an understanding of the mathematical
thinking processes (which are not reducible to
deductive reasoning, as logic is only a section of
mathematics and is only used in some stages),
and the overall picture of the field with examples of
sufficiently varied mathematical behaviours to be
acquainted with the historic movements and their
contributions. We need to learn to transfer to our
pupils the power that mathematics gives over reality,
not only the practice of some skills.

The curriculum for teachers in Universities and
Colleges must be altered to fit present needs. Our
duty to the future demands that teachers have
an insight that differs from that of scientists and technologists. We educate through mathematics, and
that means opening new vistas, taking the student to
higher levels which can no longer be equated with
Algebra, Geometry, Trigonometry and Calculus (all at
least 300-year-old subjects). We have to understand
the way that mathematicians become aware of
structures; which are the important ones; how they
can be made to act upon each other to provide classes
of relationships about which this or that can be said.
The truth of any mathematical statement depends on
the fact that everyone can ensure for himself (sic) that
the statement is contained in (is deducible from) the
set of relationships used to introduce the situation.
As teachers, we need to know how to classify the
mathematical species so that they can be recognised
and the appropriate statements made immediately.
We need to know when a situation requires that
we use topological or algebraic arguments; how to
disentangle the components of a complex entity and
take steps to improve our grasp of it.

This sort of curriculum is being developed in some
places, and it is possible to become acquainted with
it by reading, for example, Professor G. Choquet’s
course of analysis or the latest publications in the
U.S.A. We in England have little to show. What
is needed is the continuous development of a
sensible approach valid throughout the whole course
of school education, replacing the present bottle
neck resulting from all sorts of influences dating from
various times.

We need to know how to present our work to
those learning mathematics. In fact, we need a
methodology of the methodology of mathematics
teaching.

The teaching of mathematics has one aspect that
relates to the classroom, that being its methodology;
but each science has developed, besides its results,
an interest in how these results are obtained, what
the problems of the science are, how they differ from
those in others, etc. This part is the one that has, so
far, been so utterly neglected in the methodology of
teaching.

When we know why we do something in the classroom
and what effect it has on our pupils, we shall be able
to claim that we are contributing to the clarification of
our activity as if it were a science. One aspect of this,
at least, has been developed a little; I mean the study
of children’s mistakes and what we can learn from
them. This is one of the most fascinating fields of
creative work for teachers, since they are all the time
confronting the teaching reality and are the people
most interested in seeing clearly in it.

Another aspect of our work could be the study of
the mathematical possibilities of children. So much
here is sheer prejudice at present. Indeed, it is
now becoming known that children’s mathematical
abilities are incomparably greater than has previously
been suggested. Much time, energy and frustration
could be avoided if teaching were related to children’s
thinking powers.

My last point will be concerned with experimental
teaching. In our Association this theme has often
been touched upon. We do not want anybody to share
our views until these have been proved right in the
classroom. We do not think that we are entitled to
anyone’s hearing unless we have assured ourselves
in advance that we ask for his (sic) attention on the
grounds of our successful experience. If all teachers
demanded this standard of behaviour from all their
advisers, it could contribute greatly to making the
profession a responsible one, where words mean
exactly what they say. Experimental teaching is the use of the classroom in order to learn something worth communicating about the activity of teaching, so that improvements can follow in the work of others who are facing similar circumstances. In conclusion, I want to say that what matters most is that we stop being the toy of prejudice and cliché, and seriously embark upon doing our work as if it were the most precious activity we have. By becoming aware of the various components of our professional life, we shall at the same time give ourselves exciting and exhilarating moments, make a contribution to science, help the young generation to meet its future, and, more than anything, live on the side of truth.

In-sight

Shane Johnstone, MT’s artist-in-residence responds to the article, A knot theory for eight-year-olds.
Technology and communication continue to change at a fast rate which has impacted both on governance and on the ways in which members interact. Over the last year our executive committee and our business groups have moved to meeting online. A very different use of technology is the new Thursday evening activity via Twitter #beingmathematical. It was not obvious that the ethos of an ATM workshop could be achieved so successfully in this medium. I would like to thank Danny Brown for establishing such a strong way of communicating to a wide audience what our aims and guiding principles mean in practice. Danny has also set up two new ATM branches in Glasgow and Edinburgh, demonstrating how ATM is sustained by the energy of its members.

ATM has always been a mechanism for teachers to do mathematics together whilst thinking and talking about pedagogy. When we cleared out the warehouse before the move we found many handwritten, photocopied and stapled booklets that branches and working groups had written together. The wider context in which teachers work has changed radically for all but a few. The affordances to get together seem to be shrinking whilst the constraints increase. It is taking a long time for people to realise that given a serious dearth of mathematics teachers a sensible strategy is to look after the ones you have and a good way to do this is to allow time for professional development. We are conscious that it is difficult for many mathematics teachers to get to the Easter conference, and have talked for a while at General Council about having one day events. This year we have started to offer workshops based on a publication. We are extremely grateful to Miha Ollerton, Helen Williams, Anne Watson, Tom Francome and Heather Davis for taking ATM out on the road.

Another abiding concern of ATM is that mathematics is understood to be a rich creative discipline that is accessible to all. This year saw our inaugural family festival at the STEM Centre in York and we are already planning our next one. Our collaboration with the OR society who sponsored the event has been successful and we look forward to continuing to work together. We are not alone in wanting to engage the wider public with mathematics and we are keen to find and work with other groups who have similar values.

Every year is a busy one for our office team and in 2018 they have faced a wide range of challenges with unswerving commitment to ATM. Anyone who was at BCME will have met Laura, who started on a temporary basis but quickly became an indispensable part of the team. We were delighted when she accepted a permanent post. This autumn we were very sorry to accept the resignation of Kerry who has managed our office so skilfully and made a huge contribution to the restructuring of our day-to-day business. Heather Davis (GC Chair) and I talked to the team about Kerry’s replacement and their main concern was for us to find someone who “gets ATM”. When we met Sam at interview it was clear that she was just such a person. We are very pleased to welcome her to the staff team and have every confidence that whilst she will no doubt make and suggest many changes she appreciates what it means to be ATM.

Our team is good at running conferences and next year the jointly badged Easter conference is being administered by ATM. We know that we have members who want much more joint working and members who choose to sit out until it is an ATM conference. We try to balance these and it was an unusual set of circumstances that led to two collaborative conferences in a row. In 2020 we will be back to an ATM one.

When I took up a place on General Council the future of ATM was threatened by our weak financial position. Much of our focus has been on cutting costs and becoming more efficient and we owe a huge debt (!) of gratitude to Sue Pope who retired as treasurer having worked tirelessly to move us into a far stronger position.

The proposal from the MA about the longer-term future of all five of the mathematical subject associations continues to be on the agenda at the meetings of the MMSA. Our experiences of joint working both with other subject associations, and with other organisations, feed in to our discussions at our trustee meetings. One of the useful impacts of this wider debate has been to raise our awareness of the need to restructure our governance. We rely heavily on volunteers as an association and we need to make sure that we harness people's skills and enthusiasms and use their time in the most efficient way. This will be a key focus of the work of GC in 2019.

I believe the greatest strength of ATM is that our founders managed to articulate our aims and guiding principles. They underpin all our publications and the work of the MT editorial team. They enable us to devise and evaluate new activity and they are what we as trustees are trusted with. Much changes but they stay the same.

It has been a privilege to carry the honorary secretary baton and I pass it on to Louise Hoskyns-Staples knowing it will be in safe hands for the next three years.
The ATM response to the PISA 2021 consultation

by Ian Benson, convenor, functional programming and computer algebra working group.

PISA is an international assessment, carried out in April-May of Year 10 every three years. The 2021 test will major on mathematics. England participates, but does not offer any special training to participating schools. These comments are intended to inform the work of the PISA test item designers for 2021.

The main innovation in the new test is the addition of a computational thinking strand to what PISA call mathematical literacy. They also propose the systematic use of computer-based assessment of mathematics (CBAM). They say, “Mathematical literacy goes beyond problem solving, to a deeper level, that of mathematical reasoning and computational thinking, which provides the intellectual acumen behind problem solving in the 21st century”. Their conception of mathematical literacy supports the importance of students, “developing a strong understanding of concepts of pure mathematics and the benefits of being engaged in explorations in the abstract world of mathematics.” (PISA, para 20)

PISA identify the 21st century skills that mathematical literacy both relies on and develops. These are critical thinking, creativity, research and inquiry, self-direction, initiative and persistence, information use, systems thinking, communication and reflection.

The associated background questionnaire will collect information on “how classroom pedagogy is evolving on account of the impact that technology is having on our exposure to mathematics and mathematical artifacts and on what it means to do mathematics. In the case of students, it is of interest to better understand how technology is impacting on student performance which could be explored in the task performance module of the questionnaire framework. The pedagogical issues could be explored in both the learning time and curriculum and teaching practices modules.” (PISA, para 164)

The Association has a special contribution to make to the pedagogy and content required to meet the PISA objectives. ATM members have pioneered a unique pedagogy that postulates that an awareness of our somatic nature is intrinsic to developing as mathematicians (ATM, 2018). ATM members conduct research and development on a mathematics and computer science curriculum that teaches algebra before arithmetic. This helps learners develop a strong understanding of the concepts of pure mathematics (Benson and Cane, 2017). Both strands of this research build on the work of pure mathematician and educationist Caleb Gattegno, ATM’s founding Director of Studies, who proposed the teaching of algebra before arithmetic (Gattegno, 1974; Young and Messum, 2011; ATM 2017).

The Association:

• Strongly agrees with the revised definition of mathematical literacy.
• Agrees that it is important to incorporate the 21st century skills into the test items.
• Argues for a pedagogy that builds mathematical and computational thinking skills on each learner’s awareness of their mental powers.
• Recommends that textual programming languages, such as the functional programming language Haskell, are preferred to block-based languages in implementing computational thinking in mathematics. They are nearer to the syntax and semantics of conceptual mathematics. (Lawvere and Schanuel, 2004; Wells 1995)
• Warns that most of the sample test items are limited to manipulating values until a number converges on the right answer. This will not test any deep level of mathematical understanding, nor will it assess computational thinking in mathematics.
• Notes that the broadening of the mathematics curriculum to include assessment items from computational thinking to the use of simulations and dynamic geometry in computer-based assessment has implications for school IT infrastructure, teacher training and education. We believe that this investment is warranted and small compared to the scale of the expected benefits for mathematics education.

References

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Aims of ATM

The Association of Teachers of Mathematics aims to support the teaching and learning of mathematics by:

• encouraging increased understanding and enjoyment of mathematics.
• encouraging increased understanding of how people learn mathematics.
• encouraging the sharing and evaluation of teaching and learning strategies and practices.
• promoting the exploration of new ideas and possibilities.
• initiating and contributing to discussion of and developments in mathematics education at all levels.

Guiding principles

The ability to operate mathematically is an aspect of human functioning that is as universal as language itself. Attention needs constantly to be drawn to this fact. Any possibility of intimidating with mathematical expertise is to be avoided.

The power to learn rests with the learner. Teaching has a subordinate role. The teacher has a duty to seek out ways to engage the power of the learner.

It is important to examine critically approaches to teaching and to explore new possibilities, whether deriving from research, from technological developments or from the imaginative and insightful ideas of others.

Teaching and learning are cooperative activities.

Encouraging a questioning approach and giving due attention to the ideas of others are attitudes to be encouraged. Influence is best sought by building networks of contacts in professional circles.

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