When I teach beginning primary school teachers about division, I usually begin by asking them to come up with a ‘story’ for the calculation $12 \div 4$. They tend to respond, as my secondary beginning mathematics teachers do also, with a story about “sharing”. Cakes and sweets are enormously popular. This response is also typical of secondary age schoolchildren in my experience and that of the *Concepts in secondary mathematics and science* (CSMS) investigators (Hart *et al.*, 1981).

I have a suspicion that this reveals something about the mental pictures and movement schemes people carry about division. I think that the root of a significant difficulty is hidden inside this mental picture. There are two forms of division that are distinct enactively, we act them out differently when we learn about them and when we ‘do’ them in our early lives, but equivalent numerically. That is that the ‘answer’ is the same number. This is why the distinction is hidden.

The two forms of division are sometimes named partition and quotition but are more commonly referred to as sharing and grouping. Think about sharing out twelve sweets, one by one, amongst a group of four friends, “one for you, one for you, one for you, one for me”, repeated until you run out of sweets. It is similar to dealing cards. This is the division we call sharing and is the one almost all of the stories above are based on. I would keep dealing out sweets until I ran out, possibly checking near the end to see whether the resulting quotas, sweets per person, were ‘fair’ or whether there is some remainder. Remaining sweets might be cut up, to give fair shares (quotas) which are not an integer number of sweets.

The other form is grouping. It is seen in contexts like boxing up items, filling carriages on fairground rides or setting up teams for games. If I want my twelve pupils to get into groups of four, I might ask, “Albie, Bonnie, Clint and Dilyse, can you work together. Edie, Frank, George and Hannah, you can be a group,” and so on. Notice, here I pick pupils in fours. I might well be more aware that I am subtracting four each time (repeated subtraction) from my total class of pupils (my dividend). Any remainder here cannot meaningfully be split. I might need one group of three children for my group-work task.

The three quantities involved in a division each have their own proper names. Each also has its own unit. You will notice that the units also tend to show us whether a division is sharing or grouping.

Young children happily expand their vocabularies on a daily basis. I would suggest that these words could be taught to give them a language to use in order to describe sharing and grouping situations with clarity and thus make the mental distinctions and patterns clearer.

Note that in sharing the unit of the divisor is simple, children, whilst that of the quotient is compound. In this case, sweets per child.
Contrast this with grouping, where the divisor has a compound unit, pupils per group, leading the quotient to have a simple unit, the number of groups.

So far we have seen that the hidden duality of division has its roots in the distinctions between sharing and grouping. My sense is that these two aspects are rarely made explicit in early secondary mathematics teaching in England and I guess that this is also true in many primary classrooms. My conjecture is that this hiddenness contributes strongly to learners’ difficulty with division. This difficulty may be compounded by our tendency to teach algorithms early. The order of writing may confuse the learner about which is the dividend and which the divisor and also signal one of the forms of division when in fact the other form is being attempted.

For example, in sharing scenarios like £158 ÷ 5 people, the ‘bus-stop’ algorithm is often read left-to-right as “fives into 158” which linguistically signals grouping and places the dividend, the number inside the bus-stop second in the spoken phrase as opposed to first in the ‘linear’ version.

This algorithm can also be interpreted as a sharing, that is:

100 to split between five people, how many hundreds does each get, none; use the hundred to make ten tens, so now fifteen tens to split between five people giving three tens each. Finally eight ones split between five people gives one, one each and three ones remainder.

Or as a grouping:

How many fives in one (hundred; less than one (hundred fives) so call it zero and recycle that hundred, how many fives in 15 (tens); three (tens of fives); how many fives in 8, just one and three remainder.

In both cases, the algorithm relies on division being distributive over addition, allowing the dividend to be partitioned on place-value lines, another element of complexity. Chunking algorithms on the other hand tend to signal grouping. It is more obvious what the dividend is, but re-constituting the quotient may be demanding. Here I have switched to a grouping problem:

Up to now, our dividends and our divisors have all been integers, suggesting discrete data. When we introduce non-integer values, possibly from continuous data, another hurdle presents itself. For example, 158 ÷ 1.5 is hard to imagine as a sharing problem, certainly we cannot have 1.5 people.

We could perhaps think of sharing 158 apples between 1.5 wooden crates, a full-sized crate and a half-sized mini-crate, yielding 105 1/3 apples per full, whole, crate. But most often, the contexts we can imagine will involve grouping, for example 158 litres of lager ÷ 1.5 litres per stein yielding 105 1/3 full steins.

In order to understand division further, pupils need to consider the relationships which exist between dividend and quotient (direct proportion) and between divisor and quotient (inverse proportion), perhaps helped by considering the units involved. This can be linked to developing families of facts from a given trio of related quantities and can help with definitions of reciprocal to support general multiplicative and proportional reasoning.

There follows two extracts from classroom dialogues and visual representations shared by my colleague Kathryn. As you read them, can you identify which form of division problem the children were discussing? Do you think it would have been possible to teach them the terms dividend, divisor, quotient, remainder as part of these episodes?

**Example 1**

Accurate use of language is at the heart of understanding any mathematical concept. In one primary classroom, the dialogue between the teacher and pupils is around interpretation of the problem in the first instance. This example demonstrates how the pupils interpreted the problem and then discovered
the type of division they were attempting to solve through real-life situations. There was an element of role play followed by visualisation techniques to ensure understanding. The conversation around it was as follows:

Mr Green the grocer has had a delivery of 52 oranges. He has 4 empty boxes on his shelf in which to put them. How many oranges should he put in each box?

T: What kind of a problem is this?
P: It's a division one.

T: How do you know?
P: Because there are some oranges and we have to put them into four boxes, so we're going to give them out between the boxes. We have to share them out.

T: Does that seem right? Is it true? Shall we divide? Is this a sharing problem?
P: Yes, we can share them out.

T: Have we done a division like this before? Tom says that we are sharing … have we shared before?
P: Yes, when Lucy brought her sweets in for her birthday, she shared them between all thirty-one of us.

T: So, what's the same and what's different about this new problem?
P: Giving the sweets out to each person is like giving out the oranges. There are only four boxes though, but there were thirty-one children.

T: OK, so what will our equation look like? [Teacher scribed as pupil answered.]

52 ÷ 4 = □

T: Let's read the equation together.
P: Fifty-two oranges divided into four boxes equals ............ oranges in each box.

T: What does the fifty-two represent in this equation?
P: The fifty-two represents the number of oranges Mr Green had delivered at the start. That is the whole.

T: That's right. What does the four represent?
P: The four represents the number of boxes to put the oranges into. They are the parts.

T: So, what does the □ represent?
P: The □ represents the number of oranges in each box at the end.

T: How shall we solve the problem?
P: We need to start sharing the oranges.

The children proceeded to tackle the problem independently using Dienes apparatus. They started by putting single unit cubes into each box, calling each cube an orange, but very quickly one child discovered that it was slow and said, "We could put two at a time into the boxes". The teacher then guided the pupils into efficient use of the equipment.

T: Is there a way of being even more efficient? Everyone place in front of you five ‘tens’ and two ‘ones’. I can see that Amy did that. Are we able to share the tens equally between the 4 boxes?
P: Yes.

T: How many tens will each box contain?
P: One ten in each box.

T: What shall we do next?
P: There is still a ten stick left, so we need to split it into ‘ones’.

T: OK, everyone do that. We can regroup the ten. So, what can you see now?
P: There are twelve ones now. We can share them.

T: Agreed. How many ones can be put into each box?
P: Three ones. That means there are thirteen oranges in each box.

T: Are there any oranges left to share?
P: No, the boxes have an equal number of oranges with none left.

The teacher revisited the question and asked the pupils to speak the number sentence or equation again, this time including the result of the calculation.

P: Fifty-two oranges divided into four boxes equals thirteen oranges in each box.

T: What did the fifty-two represent again?
P: The fifty-two represented the number of oranges Mr Green had delivered at the start.

T: What did the four represent?
P: The four represented the number of boxes to put the oranges into.
T: What did the thirteen represent?
P: The thirteen represented the number of oranges in each box in the end.

Example 2

Problem: A baker has completed making a batch of 96 pies. Once they have cooled, he will need to put them into boxes ready to sell. Each box can hold 8 pies. How many boxes will the baker need?

The teacher observed the pupils tackling this problem on whiteboards. There was a mixture of recording strategies, such as those children attempting an answer through mental arithmetic and one pupil who used recall of division facts to record an answer. However, it was unclear how that pupil had interpreted the problem. Another method involved drawing.

T: What kind of a problem is this?
P: Another division one.
T: How do you know?
P: Like the oranges we have to share them out.
P: It’s not the same though because we have to sort out eight pies and pack them, then do it again and again until they are all gone.
T: Who is right? Are they both right? Are neither right?
P: They are both right that it is dividing, but I agree with the sorting into eights.
T: What kind of dividing is that?
P: It’s grouping, like the time we had to put doughnuts into boxes of four. Every box had to contain four doughnuts.
T: That’s right. We have to carry out the problem in a different way in real life.

What will the equation look like this time?
P: 96 ÷ 8 = 12

T: Ninety-six pies divide into boxes of eight pies equals …………… boxes needed.

Try this problem with your Dienes apparatus again, but this time, what are we calling each unit cube?
P: Each unit cube will be one pie.
T: OK, what are we trying to find again?
P: How many boxes does the baker need for his pies?

The pupils immediately started to gather nine longs (tens in this case) this time and six units. It was only when a child pointed out that the cubes in tens did not really help to create boxes of eight pies that the thinking changed. The pupils started to gather ninety-six unit cubes instead. It was apparent that the pupils were used to using the whiteboards for helping to record their mathematics because there were drawings appearing on them each time a box was full.

8 pies 8 pies 8 pies 8 pies 8 pies 8 pies 8 pies 8 pies 8 pies
8 16 24 32 40 48 56 64 72 80 88 96

T: Have we succeeded in creating groups?
P: Yes, every box is a group.
T: So, how many boxes have we filled? How many groups are there?
P: There are twelve boxes of pies, twelve groups.
T: Let’s revisit the equation. 96 ÷ 8 = 12. How shall we make sense of it?
P: Ninety-six pies divided into boxes of eight pies in each box equals twelve boxes needed.
T: Is there a way to check the calculation?
P: We can multiply the number of boxes by the number of pies in each box.
T: What is that called and what will that show?
P: It is using the inverse operation and it should tell us how many pies the baker started with.

I expect you will have found this all a bit complicated, I certainly do, but really, that is my point. We too often present division to pupils as a simple thing, just the inverse of multiplication, so we are surprised by how often pupils do not divide 5x by 5 when solving for x, or they guess what number 5 needs to be multiplied by to give 158 instead of actually dividing. In reality, there is a lot going on beneath the surface to be grappled with. As a teacher, I need to grasp these ideas so that I do not present mathematics to my pupils in a contradictory and confusing way. As a pupil, a better grasp of these ideas, through this language, will help me to reason with and to comprehend some of the most fundamental concepts in secondary maths.

References

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