Problem solving, mastery and variation: The acoustic version

A group of mathematics educators discuss how the English National Curriculum aim of developing problem solving fits with the use of the language of ‘mastery’.

We call this paper “the acoustic version” because we are trying to strip away some of the trappings that have accumulated around mathematics teaching since the 2013 curriculum was introduced, and get back to what we see as basic ideas about mathematics teaching and learning, namely the purposeful choice of examples and the development of a problematising mindset towards mathematics.

Mastery

For the purposes of this discussion, we take mastery to mean conceptual understanding and fluency. That is, in our words:

- a well-founded accumulation of successive layers of conceptual understanding and experience so that mathematical structures can be recognised in situations, both within mathematics and in outside contexts, to which knowledge and methods can be applied both to solve problems and also to make further progress in understanding.

Thus, we are taking the achievement of mastery to be an ongoing quality of learning. Mastery does not describe a particular method of teaching, and no method guarantees mastery. We developed the above statement between ourselves as a working, shared, understanding. For more discussion on this we refer you to a paper produced by the National Association of Mathematics Advisers, debunking Five myths of mastery in mathematics. They emphasise that:

- mastery does not have one clear definition.
- mastery does not prevent differentiation.
- mastery is not a special curriculum.
- mastery does not imply repetitive practice.
- mastery does not mean you have to use particular textbooks. (NAMA 2016)

Nevertheless, qualities of teaching are central to the achievement of mastery and our discussion is about what pedagogical actions make this more likely for more students. We use three situations to illustrate our discussion:

1. A question from a KS2 test: if \( a + b = 20 \) and \( a - b = 7 \), what are \( a \) and \( b \)?

2. A ratio question observed in a lesson: Suki and Muki share £20 in the ratio 2:3. How much do they each get?

3. The move from understanding linear functions to quadratic functions.

In each situation, we ask, “What is needed, in terms of variation and problematising, to tackle these questions or make these moves successfully”? Our aim is to suggest teaching approaches for progress towards mastery, in the way we have defined it.

If \( a + b = 20 \) and \( a - b = 7 \), what are \( a \) and \( b \)?

This test question is inaccessible if you have no experience in using letters and expressions to represent arithmetical situations. Therefore, these need to be familiar to you and you need to be able to read algebra meaningfully. Solving simultaneous equations is not in the English primary national curriculum, so there is no standard expected method. Instead, you have to use problem-solving skills and experience to explore some possibilities and not be put off when you cannot find any whole numbers that work. It helps to reflect on the whole numbers you have tried and see if any of them give clues about what else you might try. Another approach might be to think about where a might be placed on a number line, so that 20 and 7 are equal distances away from it. A further approach might be to coordinate two part-whole diagrams. In more general terms, you have to approach this as a problem to be explored and solved and not as if there is a method to be applied. You might have to try some numbers systematically, or think of a representation that might give you some insight, and you might have to adapt any initial assumptions about whole numbers. Without mastery of number bonds up to 20, and mastery of fractions or decimal parts of whole numbers and the ability to read simple algebra, you could be stuck. Here, mastery must also include the kind of fluency that brings crucial ideas to mind when they are appropriate. Equally you would also be stuck without a problem-solving mindset and some experience of exploration, adaptation and representation. Whether being fluent in these qualities of doing mathematics is seen as a component of mastery or not depends on the definition, but to work on questions like this it helps if
students have a repertoire of useful representations, fluent number facts and are enculturated into problem-solving habits throughout school.

Suppose that instead of this being a test question, you used it as a lesson task. Would you start with $a + b = 19$, rather than 20, since that is soluble with whole numbers? Would you vary the sum so that students could see why thinking only about whole numbers limited them? Would you encourage a diagrammatic approach, or a rod-based approach, to show that $a$ is the mid-point of an interval of length $2b$ (see diagram)? What would you do about the fact that, as you vary the sum, some solutions are easy to find and some are trickier if you are locked into an integer world? Would you classify those that are easy and those that are harder, and discuss why this is so?

This question, presented with variation of the sum (or the difference), provides fertile ground for reasoning which could be inductive from several examples, or could draw deductively on previous knowledge of odds and evens (if the sum and difference are both even, or both odd, the answers are whole numbers; if not what does this imply?). Furthermore, it provides grounds for conjectures and even some reasoning towards a generalisation: $2a = \text{sum} + \text{difference}; 2b = \text{sum} - \text{difference}$. We can imagine further variations and explorations of what is the same and what is different. We can also recognise the beginnings of powerful methods for solving simultaneous linear equations later on, for which fluency will be useful. But knowing particular methods for tackling this particular problem is not the point; progress towards mastery is supported by the collection of a toolkit for future use from which an expert would choose appropriate tools for an unfamiliar task.

**Suki and Muki share £20 in the ratio 2:3**

Students were observed in a lesson working with this question, using toy money. Eventually, after dealing out some pounds, they realised that one person was going to have two of something and the other would have three of something, so the ‘something’ would be whatever gave you 20 when you multiplied by 5. After further effort, they found out that the unit, or part, was £4. The problem was solved, but the final step had taken so long that the important part of the process, that is adding the parts to get the total number of parts, had become lost in the midst of lengthy calculation time. Solving this one problem was not going to give them the experience of a method that could be used in other, similarly structured, contexts. This is a situation in which technical fluency would be valuable, since ratio questions are likely to appear again and again in a student’s mathematics lessons. So, how can technical fluency, i.e., remembering the method, be achieved? Variation theory indicates that random variation of numbers does not generate a strong focus on the critical idea, that you have to work out the total number of parts and then divide. Controlled variation allows this to be seen through inductive reasoning, or through enactive similarity, while also leading to understanding why it must be so. Suppose the ratio is kept at 2:3 but the amount is changed to £30, and then £45 and so on. That variation would focus on the division so long as students knew multiples of 5 and did not get bogged down in calculations. Or suppose the total is kept at £20 but the ratio is varied to 1:4; 7:3; 1:3; and so on. That variation would focus on the need to sum the parts as a generalised action, so long as students could do the division. For each of these kinds of variation, a discussion about what has changed and what has stayed the same would emphasise the role of the different numbers involved. If they cannot do the division, all the effort goes into the calculation rather than reflecting on the method. If students understand why they are adding the parts, they might eventually work with other numbers and with triple or quadruple ratios and explain why their methods work.

To get to that layer of understanding they need to be focusing on adding the parts, moving on from adding toy money while making light of the calculations. This is different from teaching that allows some students to stay with adding toy money, and others to spend class time working out the relevant multiplication facts. To avoid the former sticking point, diagrams and imagery can provide a bridge to abstraction; to avoid the latter sticking point, the teacher can anticipate what might happen and prepare students before the lesson in extra time, or provide necessary aids. Mastery of the key idea depends on it being the central and most obvious aspect of the lesson for all learners, rather than merely getting answers, so public talk about adding the parts would be an important ingredient of teaching.

Depending on which versions you read, variation theory suggests that the teacher can:

- vary the critical aspects of what it is hoped
students will learn. Here that would mean varying the parts and keeping £20 the same so that adding the parts becomes the common experience.

- offer several methods for solving the problem. Here that might mean providing toy money, counters, a number line or pencil and paper.

- vary the way you pose the question. Here that might mean saying, “when sharing out some money unequally, Suki gets £8 and Muki gets £12. Write this as a ratio in simplest form and say what size each part would be.”

- when adding parts is secure, vary the total and the ratios at the same time, but without making the calculations significantly harder. This is probably not the right place to practise dividing by 17.

**Move from understanding linear functions to quadratic functions**

Here we discussed several approaches, each one resting on assumptions about what students already understand and each one focusing on a different aspects of quadratics. From a variation perspective, the key questions would be:

How do quadratics differ from linear functions?

What aspect of quadratics shall we focus on?

What examples draw attention to that aspect?

In these approaches, quadratics appear as variations to features of linear functions that students have met before. Before we describe these, it is worth thinking about how linear functions might have been introduced to them. Variation theory suggests that these too could have been introduced as a variation of something already known, maybe (i) as a special kind of sequence that is generated by adding or subtracting the same amount successively, or (ii) as a special class of graphs that are straight lines or (iii) as representations of situations of constant rate, such as gradient, density, price per unit, that is, they have had some mathematical or realistic purpose or (iv) as algebraic structures. Each of these possibilities gives different opportunities to approach quadratics using the question, “What is the same and what is different?”

(i) Quadratics can be introduced through posing questions about quadratic sequences, but sequential approaches run the risk of generating a pointwise view of functions and assumptions about interpolation that may not be justified. For example, just because a function includes the ordered pairs: (1, 2), (2, 4), (3, 6) does not mean it is linear, and similar faulty reasoning can apply to quadratics such as (1, 2), (2, 5), (3, 10). However, this approach would draw attention to varying gradient, because the differences between the y-values change, so we have varying rate of change. We can still talk about average rate of change over an interval.

(ii) Quadratics can be introduced as graphs that are not straight lines, but that is not precise enough. Rather, one aspect of linear functions could be chosen to vary, such as intersecting with the x-axis in two, one or no places, whereas a linear function intersects it in one or no places. This is still not precise enough, but uses variation to pose questions. Seeing quadratics with real roots as products of two linear factors could help, e.g., \( y = (x - 1)(x + 2) \) rather than \( y = x^2 + x - 2 \).

(iii) Quadratics can be introduced as patterns for some curved objects, a variation from the slopes and staircases that may have been used to introduce linear functions. For example, designing a range of winelgases using a graph plotter to draw quadratic curves, and being asked to vary a, b, and c in \( y = ax^2 + bx + c \) to get curves suitable for different uses. This has the advantage of focusing students on the shape of the whole function, scaling in two directions and varying gradient, rather than a pointwise approach. Scaling linear graphs alters the slope, but not the shape; scaling quadratics changes the shape.

(iv) Another approach assumes that students have spent some time becoming fluent with the algebraic representation of linear functions, i.e., \( mx + c \), and how this relates to the properties of the function. To encourage a shift towards thinking of functions as objects in their own right, students can be asked to imagine what \( y = x^2 + 3x + 2 \) might look like. Using their existing knowledge of \( y = 3x + 2 \) they can suggest how adding this term to \( y = x^2 \) will change the function. Will it change the y-axis intercept? Will it change the gradient?

Each of these approaches supports the accumulation of conceptual understanding and experience of (and mastery of) functions in the school curriculum more generally, as well as the specifics of quadratics. Each extends knowledge from the specifics of linear functions into (though not very far into) a more general space of functions. Each focuses on one particular kind of variation from linear functions, rather than trying to zap the whole lot at once as if quadratics
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is a fresh topic with its own set of mysteries, unconnected with anything else other than algebraic manipulation. Each calls on a problem-solving approach involving inductive reasoning from several carefully chosen examples, laying the groundwork for deductive conjectures about, for example, real roots, sequences whose differences are themselves linear and polynomial functions of order 2. Each encourages enquiries into what is the same and what is different.

Reflections on three situations

In each of these situations, we are thinking of mastery as an ongoing developmental process throughout school; not as a recipe for organising mathematics teaching. The level of conceptual detail in each of our cases does imply a need for extra preparation and follow-up for some students from time to time. The original use of “mastery” (see Bloom, 1974) to describe a particular teaching approach makes it clear that some people may learn some aspects more slowly than others, so need more teaching time to compensate so that equity of access to the curriculum can be achieved. In each situation, we show how variation of examples can be used to stimulate reasoning towards a conceptual generalisation. We also suggest that so-called new topics can be introduced as variations on previous experience, and hence treated as problems to be solved. We also show how a problem-solving mindset can be promoted and enhanced through use of variation. We have not been focusing only on what can be achieved in particular lessons or task sequences, but on what can also be influenced with overall coordination of mathematical experience within schools.

References


The members of the group of mathematics educators are Anne Watson, Nick Andrews, Helen Drury, Jenni Ingram, Sue Lowndes, Jude Stratton, Gabriel Stylianides, brought together by their association with the Mathematics Education Research Group at the University of Oxford.

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