Can game theory techniques can be used to model classroom interactions?

Rebecca Hardy uses a Scandinavian journal article, academic literature and creates an extensive form game theory model for questioning in the mathematics classroom.

What is game theory?

Created by mathematician John Nash in 1925, game theory is the branch of mathematics that explores whether logical conclusions can be drawn from situations, which are modelled as ‘games’. Games require: a set of players, a set of actions for these players and a set of outcomes, or payoffs, for the actions. The simplest game type, a normal form game, is displayed as a ‘payoff matrix’. The matrix can then be analysed using logic and mathematics to determine what conclusions can be drawn, and which strategies should be employed by the players to achieve the best results to ‘win’ the game.

Games can be classified as:

- **those of pure cooperation**
  
  The payoffs for both the players are the same and model either mutual gain or mutual loss.

- **those of pure competition**
  
  Also known as zero-sum games because the payoffs for the players sum to zero, these games model a situation where one player’s outcomes directly opposes the other.

- **a hybrid of the above game styles**
  
  Sometimes the players wish to cooperate and other times they are in competition, depending on the strategies adopted throughout the game.

Modelling classroom interactions using game theory

An article in a Scandinavian education journal analysed classroom interactions

... as though they comprise a repeated Prisoner’s Dilemma game between two players, teacher and students, where the interests of the teacher and students are often partly concurrent and partly in conflict. (Elstad 2002, 71-72)

The Prisoner’s Dilemma is a notorious game, which models a situation where two prisoners who committed a crime together must confess or defect their involvement. In this educational situation, confessing means cooperating and defecting means non-cooperation.

<table>
<thead>
<tr>
<th></th>
<th>Teacher cooperates (C)</th>
<th>Teacher does not cooperate (D)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Student cooperates (C)</td>
<td>-1,-1</td>
<td>-4,0</td>
</tr>
<tr>
<td>Student does not cooperate (D)</td>
<td>0,-4</td>
<td>-3,-3</td>
</tr>
</tbody>
</table>

Figure 1: Payoff matrix for repeated Prisoner’s Dilemma game between teacher and students

When analysing a game, it is necessary to look for any dominant strategies, meaning that a player should play this action throughout the game. Considering all the combinations that teacher and student can play, and referring to the payoff matrix, see Figure 1, the following conclusions can be drawn:

- If student plays C, teacher should play D
- If student plays D, teacher should play D
- If teacher plays C, student should play D
- If teacher plays D, student should play D

CD (-4, 0): 0 is a better payoff than -1

It is interesting to note that strategy CC is redundant, using synoptic analysis of each player’s individual choices. Taking a panoptic view of the whole matrix, strategy CC is strictly better than strategy DD, as (-1, -1) gives better payoffs than (-3, -3). Responding intuitively, both the teacher and the students should always cooperate to get the most out of the interaction. Surely the teacher should always play the strategy of cooperating, regardless of whether the students wish to or not? Analysis of the individual players’ payoffs does not support this argument. Therefore, there are flaws in Elstad’s choice of model.

Hardy’s model

Developing Elstad’s work, classroom interactions are restricted to questions. A normal form model cannot be used for these types of interactions, as questioning requires one person to ask a question, followed by a brief period where the other person registers the information and formulates an answer, before responding. In order to take into account this ‘wait time’, an extensive form game model must be used. Extensive form games do not allow players to play actions simultaneously, they must take turns. Examples of situations modelled in this way are chess and poker. Instead of a matrix, a game tree is used to analyse the game, to show how one player’s action choices leads to the other player’s actions and payoffs.

According to Mason (2010), there are several different types of questioning in the mathematics classroom, including:

- **Enquiry-questions**: a pupil has a gap in their knowledge and wants to find something out.
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- **Controlling questions**: used by the teacher when asking questions to specific pupils, in order to highlight their control and power in the classroom.

- **Cloze technique**: teachers pause at the end of a sentence and make the children fill in the missing word.

- **Genuine enquiry questions**: the teacher is genuinely interested in how a pupil completed their work, but the child assumes that if the teacher is asking them about their work, they must have done it wrong. This can lead to the pupil being defensive.

- **Meta-questions**: the teacher guides pupils to ask questions for themselves about their work, through reflection, known as “scaffolding”.

- **Open and closed questions**: closed questions require a single correct answer, whereas open questions allow the learner to choose how much information they wish to disclose in their answer and there is no unique, correct response.

Mason (2010, p. 4) makes an interesting point about open and closed questions:

> . . . questions are just words with a question mark: the notion of openness and closed-ness is more to do with how the question is interpreted than with the question itself.

If something requires a sense of interpretation, *behavioural game theory* would need to be researched and applied. Behavioural game theory explores players' thought processes when choosing which actions to play. However, using this would over-complicate the aims of the paper.

Hardy’s model is based on Mason’s questioning types to produce a complete information extensive form two player game. It is not possible to use an incomplete information game, as this suggests players do not know what the other person has asked in the past. How can a pupil respond if they do not know what the question is? In this game, the teacher always plays first, asking a question to a pupil. In some situations expressed in the model, the teacher has the opportunity to ask another question. This relates to the literature. The game tree of Hardy’s model, showing players, actions and payoffs can be seen in Figure 2.

**Players**
- T = player 1, teacher
- P = player 2, pupil

**Actions**
- ‘GE’ = genuine enquiry question
- ‘Con’ = controlling questions
- ‘Clz’ = Cloze technique
- ‘Met’ = meta-questions
- ‘l’ = let them off
- ‘p’ = punish them
- ‘h’ = help
- ‘n’ = don’t help
- ‘a’ = answer
- ‘d’ = don’t answer

**Payoffs**

The numbers are arbitrary choices, created from the ideas posed in the literature.

> The key to effective questioning lies in rarely using norming and controlling questions, in using focusing questions sparingly and reflectively, and using genuine enquiry-questions as much as possible.  

(Mason 2010, p12)

Greater onus is placed on genuine enquiry questions. If a teacher asks a genuine enquiry question that a pupil answers, both players receive a numerical payoff of 3. If the pupil does not answer the question, the teacher receives a payoff of 0, as they have no new knowledge of the student’s thought processes. The pupil receives -1, as . . . the respondent may interpret the question as an indication that there is something wrong . . . [and are] likely to take a defensive stance.  

(Mason 2010, p3)

The worst payoffs for both players are for controlling questions or the cloze technique, as these questioning styles do not provide enough scope for a child to really think mathematically. Therefore the maximum payoff for both teacher and pupil when these question types are used, and the pupil responds, is 1. However, there are also cases where a pupil does not answer these questions.

If a pupil does not answer a controlling question and is ‘punished’, their payoff is -1, but if they are ‘let off’ for not answering, their payoff is 1. The teacher receives a payoff of 0 in both these scenarios, as they do not get an answer to their question.
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In the case of the cloze technique, where the child does not answer, both teacher and pupil receive a payoff of 0.

As meta-questioning is effectively scaffolding learning, the teacher is guiding the pupil to think for themselves through the use of the chosen question. Therefore, if both players cooperate, the maximum payoff is 2 for both teacher and pupil. If the child does not answer the original question, both players receive a payoff of 0 and neither party gains anything from the interaction. For the scenario where a pupil answers the original meta-question, but the teacher fails to scaffold their learning by not asking another question; or the scenario where teacher does scaffold the learner, but the child does not answer this second question; both teacher and pupil will receive a payoff of 1. This is because only one player is cooperating.

Analysing the game

The extensive form tree created can be converted into a normal form matrix, in order to analyse the game and calculate the Nash equilibrium. The Nash equilibrium shows whether the game is ‘stable’. First, the pure strategies for each player must be found.

For the pupil, there are five equal choice nodes of size two. This means the total number of permutations (choices of actions) is $2^5 = 32$. These are:

- aaaa, aaad, aada, aadaa, daaa, aaadd, adad, ada, ad, ada, add, dada, dadaa, adddd, adda, dada, dadaa, add, adaad, adddd, adda, add, dadaa, dadaa, dada, dada.

For the teacher, there is one choice node of size four and two choice nodes of size two. This means the total number of possible pure strategies is:

\[4 \times (2^2) = 4 \times 4 = 16,\]

which are:

- Ge l, Ge p, Ge n, Con l, Con p, Con h, Con n, Clz l, Clz p, Clz h, Clz n, Met l, Met p, Met h, Met n.

These pure strategies are the actions to be written along the top and down the left-hand-side of the matrix table. It is clear that the matrix relating to the full tree is going to be very large: a table of 16 by 32.

Sub-game 1: Genuine enquiry questions vs. controlling questions

In order to calculate the payoffs, it is necessary to simulate playing the game. Graphically, this means highlighting the lines that show the possible actions that could be played during a turn. The action choices that create a complete path from the first choice node (the first question asked by the teacher) to a terminal node (where the payoffs are written) give the payoff to be taken.
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of dominant strategies was followed. This means exploring whether some actions should be played over others, and iteratively removing these dominant strategies until none remain dominant. In this scenario, there were no actions that should be played over others. This means that the game can be analysed no further. Therefore, no conclusions were drawn.

**Sub-game 2: Genuine enquiry questions vs. Cloze technique**

![Diagram of game tree for sub-game 2](image)

Figure 7: Tree for sub-game 2

Again, the payoffs for each combination of pure strategies are found, see Figure 8.

<table>
<thead>
<tr>
<th></th>
<th>aa</th>
<th>ad</th>
<th>da</th>
<th>dd</th>
</tr>
</thead>
<tbody>
<tr>
<td>GE</td>
<td>3,3</td>
<td>3,3</td>
<td>0,-1</td>
<td>0,-1</td>
</tr>
<tr>
<td>Clz</td>
<td>1,1</td>
<td>0,0</td>
<td>1,1</td>
<td>0,0</td>
</tr>
</tbody>
</table>

Figure 8: Payoff matrix table for sub-game 2

Using the method of the iterated removal of dominant strategies and Figure 8, it is clear that strategy **aa** strictly dominates **dd**, as the payoffs for strategy **aa** are always better than those of **dd**.

Note: **ad** and **da** are only weakly dominated by **aa** so these cannot be removed.

\[
aa \geq ad: 3=3 \text{ and } 1>0 \quad aa \geq da: 3>-1 \text{ and } 1=1
\]

Removing strategy **dd** from the normal form payoff matrix produces Figure 9.

<table>
<thead>
<tr>
<th></th>
<th>aa</th>
<th>ad</th>
<th>da</th>
</tr>
</thead>
<tbody>
<tr>
<td>GE</td>
<td>3,3</td>
<td>3,3</td>
<td>0,-1</td>
</tr>
<tr>
<td>Clz</td>
<td>1,1</td>
<td>0,0</td>
<td>1,1</td>
</tr>
</tbody>
</table>

Figure 9: Payoff matrix for sub-game 2, excluding strategy **dd**

Using Figure 9, it is possible to calculate the mixed strategy Nash equilibrium for the game. This method of analysis allows the players to determine how often they should play each of their remaining optional strategies, in order to receive the highest payoffs throughout the game.

<table>
<thead>
<tr>
<th></th>
<th>P</th>
<th>R</th>
<th>1-P-R</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>Q</th>
<th>GE</th>
<th>Clz</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3,3</td>
<td>1,1</td>
</tr>
<tr>
<td>1</td>
<td>0,0</td>
<td>1,1</td>
</tr>
</tbody>
</table>

Figure 10: Matrix with algebraic characters for the strategies

For the pupil:

\[
3P + 3R + 0(1-P-R) = 1P + 0R + 1(1-P-R)
\]

\[
3P + 3R = P + 1 - P - R
\]

\[
3P + 4R = 1
\]

Therefore, \( P = \frac{1}{6} \) and \( R = \frac{1}{8} \). This means that the **pupil should play strategy aa \( \frac{1}{6} \) of the time, strategy ad \( \frac{1}{8} \) of the time and strategy da \( \frac{17}{24} \) of the time.**

For the teacher:

\[
3Q + 1(1-Q) = aa
\]

\[
3Q = 1 - 2Q
\]

\[
Q = \frac{1}{5}
\]

Therefore, the teacher should play strategy **GE** \( \frac{1}{5} \) of the time and strategy **Clz** \( \frac{4}{5} \) of the time.

These outcomes show that in order to get the most out of interactions, the teacher should be using the **cloze technique more often than genuine enquiry questions**. This does not agree with the literature. For a pupil, the most popular strategy to use is don’t answer, answer. Answering both questions is the least popular strategy. This could suggest that, in general, these questioning types do not lend themselves to cooperation from a pupil.
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Sub-game 3: Genuine enquiry questions vs. Meta-questions

Figure 12: Tree for sub-game 3

<table>
<thead>
<tr>
<th></th>
<th>aaa</th>
<th>aad</th>
<th>ada</th>
<th>add</th>
<th>dda</th>
<th>dad</th>
<th>dda</th>
<th>ddd</th>
</tr>
</thead>
<tbody>
<tr>
<td>GE h</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>0,-1</td>
<td>0,-1</td>
<td>0,-1</td>
<td>0,-1</td>
</tr>
<tr>
<td>GE n</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>3,3</td>
<td>0,-1</td>
<td>0,-1</td>
<td>0,-1</td>
<td>0,-1</td>
</tr>
<tr>
<td>Met h</td>
<td>2,2</td>
<td>1,1</td>
<td>2,2</td>
<td>1,1</td>
<td>2,2</td>
<td>2,2</td>
<td>1,1</td>
<td>1,1</td>
</tr>
<tr>
<td>Met n</td>
<td>1,1</td>
<td>1,1</td>
<td>0,0</td>
<td>0,0</td>
<td>0,0</td>
<td>1,1</td>
<td>1,1</td>
<td>0,0</td>
</tr>
</tbody>
</table>

Figure 13: Payoff matrix table for sub-game 3

The only dominant strategy is aaa, whose payoffs are strictly better than ddd. Some other strategies weakly dominate each other, for example aaa is weakly better than add and ada is weakly better than dda, but these cannot be iteratively removed. After removing strategy ddd, there are still 7 choices for player 2. It can be seen that none of the teacher’s strategies are fully strictly dominant either. However, looking at the situation where a pupil answers the first question, the teacher’s GE h and GE n strategies will dominate the others.

This shows that if a pupil is going to answer the first question, then it is strictly better for a teacher to ask a genuine enquiry question. The counter argument cannot be used, where the teacher should ask a meta-question if the pupil does answer the first question, as the payoffs are not strictly better for Met n against GE h.

An interesting thing to note here is that the teacher’s only dominant strategy is based on a pre-understanding of how the pupil will answer.

Conclusions

Although the literature advised that genuine enquiry questions be used “...as much as possible” (Mason 2010, p12), analysis of the three sub-games from Hardy’s model showed this to be false.

When compared to the Cloze technique, it was seen that for every 5 questions asked, genuine enquiry questions should be used only once.

When compared with meta-questions, it was seen that in order for teachers to use genuine enquiry questions more often, they must hold a predisposition that the pupil will definitely answer the question.

Overall, it is difficult to determine which questioning types should be adopted in order to get the most cooperation from pupils. Therefore, game theory may not be appropriate to use for modelling questioning in the classroom.

Areas for further exploration

Only three sub-models were explored from Hardy’s model. Different combinations of questioning types could be analysed using game theory to determine if some styles of questioning produce the best outcomes. As well as this, no conclusions could be drawn, using game theory techniques, whether genuine enquiry or controlling questions were better for teachers to use. Therefore, further exploration of strategies for questioning in the classroom can be examined. This would also be useful for proving whether game theory is a useful tool strategising questioning in the mathematics classroom.

Rebecca Hardy completed this study in her final year, 2013, at Sheffield Hallam University

References


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