Alan Beardon’s Problem

How many integer-sided triangles are there with a specified integral in-radius?

This was one of the challenges Geoff Faux issued in the opening session at the 2013 conference. His written account contains these words: “I would like to pass on a ‘wondering’ from Alan Beardon: Is there anything known about integer triangles round a circle of radius 2?”

The following pages chart the course of the investigation which followed.

Colour key:
Paul’s e-mails are shown like this.
Paul’s e-mails attachments are shown like this.
Alan’s e-mails are shown like this.
Paul addresses you, the reader, like this.

Dear Geoff,

07.39, 24.11.13

Having received my copy of MT237, it was a pleasure to re-engage with the problems the Pythagoreans had in designing their universe. This is as far as I’ve got with Alan Beardon’s problem.

As you see, the first section, ‘Preparation’, is just me working through the right-hand column on p.11 in my own terms. Compare the second section, ‘The breeding programme’, with the notes exchanged between yourself and John Mason.

Best wishes, Paul

Preparation

With no requirement on the sides, there would be an infinite number. (Construct 2 tangents to the circle and let a third roll round it. An infinite number of positions and therefore cut-offs is possible.)

With the stated requirement we need only consider rational sides and scale up as necessary.

Let the triangle have sides a, b, c, altitudes h_a, h_b, h_c and semiperimeter s.

We express the area in 3 ways:

\[
\frac{ah_a}{2} \text{ etc. (1)} \quad rs \quad (2) \quad \sqrt{s(s-a)(s-b)(s-c)} \quad (3)
\]

From (2), (3) we have

\[
r = \frac{(s-a)(s-b)(s-c)}{s} \quad (A)
\]

From (1), (2) we have

\[
r = \frac{h_a h_b h_c}{h_a h_b + h_b h_c + h_c h_a} \quad (B)
\]
(B) tells us that rational \( r \) requires rational altitudes and, conversely, rational altitudes guarantee rational \( r \).

(A) tells us that rational \( r \) requires rational sides but the square root sign tells us that this is not enough. We therefore require both rational sides and rational altitudes, viz. a heronian triangle.

If we could identify and list all heronian triangles with greatest side \( \leq N \), we would have all rational in-circles in triangles with greatest side \( N \).

Consider this triangle:

Since \( h_a \) and \( c \) are rational, the Pythagorean condition ensures that \( a_c \) is too. Likewise for \( a_b \). Every heronian triangle can therefore be made by the geometric addition and subtraction of Pythagorean triangles.

We note in passing a stronger property of Pythagorean triangles.

Consider this primitive Pythagorean triangle:

\[
c = a + b - 2r, \quad \text{whence} \quad r = \frac{a + b - c}{2}.
\]

Since \( c \) must be odd, and one of \( a, b \) must be odd, the numerator must be even, whence \( r \) must be integral. - Is there one (or more) Pythagorean triangle(s) for every \( r \)? (The right-hand figure shows how Liu Hui (early C3) dissects the triangle to derive the formula. He is commenting on problem 16 in chapter 9 of the Jiu Zhang Suan Shu (‘the Chinese Euclid’).)

The breeding programme
The length of one altitude in the new triangle is the LCM of a leg in each of the right triangles we choose. For every triangle we obtain by addition, we obtain a complementary one by subtraction. (Congruent right triangles constitute a trivial case: annihilation. Adding/subtracting nothing is the other trivial case.)

Say we’ve found all the triangles with in-radius 1 to \( r \). The triangles with radius \((r + 1)\) will comprise:

- all those whose in-radii are factors of \((r + 1)\), scaled up,
- all those freshly discovered with rational but non-integral radius \( \frac{r+1}{k} \), scaled up by \( k \),
- all those freshly discovered with integral in-radius \((r + 1)\).

Let \( n(r) \) be the number of triangles with in-radius of length \( r \). We would expect a big jump in moving from \( r \) to \((r + 1)\) if the latter is a ‘highly divisible’ number, especially if \( r \) is prime. For example we would expect \( n(12) \) to be much bigger than \( n(11) \). These are the values I’ve found so far. I show later what ‘\( k \)’, ‘\( l \)’, ‘\( m \)’ are and establish that the \( n(1) \) solution is unique.

\[
n(1) = 1, \text{ viz. } (k,l,m) = (3,4,5) \quad (1,2,3) \\

n(2) = 5, \text{ viz. } (k,l,m) = (3,4,5) \quad (6,8,10) \quad (5,12,13) \quad (2,4,6) \quad (2,3,10) \\
\]

\[
(8,15,17) - (3,4,5) \times 2 = (9,10,17) \quad (1,8,9) \\
(20,21,29) - (3,4,5) \times 5 = (6,25,29) \quad (1,5,24) \\
(3,4,5) \times 4 - (3,4,5) \times 3 = (7,15,20) \quad (1,6,14) \\
\]

\[
n(3) = 10, \text{ viz. } (k,l,m) = (3,4,5) \quad (9,12,15) \quad (8,15,17) \quad (7,24,25) \quad (3,6,9) \quad (3,5,12) \quad (3,4,21) \\
\]

\[
(3,4,5) \times 4 - (5,12,13) = (11,13,20) \quad (2,9,11) \\
(9,40,41) - (3,4,5) \times 3 = (15,28,41) \quad (1,14,27) \\
(12,35,37) - (3,4,5) \times 4 = (19,20,37) \quad (1,18,19) \\
(33,56,65) - (3,4,5) \times 11 = (12,55,65) \quad (1,11,54) \\
(3,4,5) \times 6 - (5,12,13) \times 2 = (8,26,30) \quad (2,6,24) \\
(8,15,17) \times 3 - (3,4,5) \times 8 = (13,40,51) \quad (1,12,39) \\
(3,4,5) \times 2 - (3,4,5) \times 2 = (10,10,12) \quad (4,6,6) \\
\]

As you, Geoff, imply on p. 12, putting an upper bound on \( n(r) \) is not likely to be an easy matter.

At this date (5.5.14) Alan is slowly bringing that upper bound down.

More integers
Taking a rest from counting, and inspecting what you’ve got, you find that three – perhaps more? - integer properties of the component Pythagorean triangles carry over into the composites:

1. *s is an integer. Proof:*

As we have seen, a Pythagorean triangle has two odd sides and one even, one of the odds being the hypotenuse. When we splice them together, these 5 situations arise. The red number shows the parity of the original; the green number, the parity of the multiplying factor; the black number, the parity of the result.

Either we have 2 odds and an even, or 3 evens. Either way, the total is even and the semi-sum, *s*, an integer therefore.
2. \(k, l, m\) are integers. Proof:

We have

\[
\begin{align*}
a &= k + l \quad \text{(I)} \\
b &= k + m \quad \text{(II)} \\
c &= l + m \quad \text{(III)}
\end{align*}
\]

whence \(c - a + b = 2m\).

But 2 of \(a, b, c\) are odd. Therefore any combination \(\pm a \pm b \pm c\) will be even.

So \(m\) is an integer (and similarly for \(k, l\)).

Though clear from the algebra, I don’t think this result jumps out of the geometry - or does it? (Notice incidentally that \(k, l, m\) are the radii of circles centred on the vertices and meeting on the sides.) The form of the set of equations (I), (II), (III) is familiar from recreational mathematics as an arithmogon, set up like this - Because it’s significant here, I’ve added a central box for \(k + l + m = \frac{a+b+c}{2} = s\):
Give your children our list of triangles and see how quickly they get the \((k,l,m)\)s from the \((a,b,c)\)s.

3. Formula (D)

Expression (I) for a right triangle with hypotenuse \(c (= l + m)\) is \(\frac{ab}{2}\).

Using (II) again gives \(ab = 2rs\), i.e.

\(ab = (a + b + c)r\). Substituting from (I), (II), (III), we have

\((k + l)(k + m) = 2(k + l + m)r\) \hspace{1cm} (C)

For all heronian triangles, whether right or composite, we substitute in (A) from (I), (II), (III), to get

\(klm = (k + l + m)r^2\) \hspace{1cm} (D)

The algebraic symmetry of (D) reflects the democracy of the sides.

I now need to justify the fact that I didn’t put a ‘+’ sign in front of \(n(1)\).

The following proof is contrived, to say the least. Can readers do better?

\((D)\) for \(r = 1\) has LHS \(klm\) and RHS \((k + l + m)\).

If any 2 of \(k,l,m = 1\), LHS < RHS.

If any 2 of \(k,l,m = 2\) or more, LHS > RHS.

Therefore no 2 of \(k,l,m\) can be equal, i.e. \(k,l,m\) must be distinct.

The smallest such set is \((1,2,3)\), giving equality.

If any greater number is substituted for any of \(k,l,m\), LHS > RHS.

Thus \((1,2,3)\) is the only solution set for \((k,l,m)\), and \((3,4,5)\) for \((a,b,c)\).
Dear Alan,

15.42, 14.12.13

... I sent Geoff the attached ... It ends, as you see, with my clumsy exhaustion proof that the 3-4-5 triangle is the only integer-sided triangle with in-radius 1. However – and this is the point of my e-mail – he tells me you have “a neat algebraic proof”. May I share this – and indeed any other observations you care to make – with the readers?

Dear Paul,

12.46, 16.12.12

I don’t have a record of my correspondence with Geoff, but I think the attached .pdf file summarises my ideas on the subject. ...

There were, you’ll realise, two (connected) deficiencies in my own draft:
A. I could offer no upper bound to n(r).
B. My ‘breeding programme’ was not a program. I had nothing to give the computer. Conceivably it could:

1. take the length of the side common to the two right triangles,
2. split it into two factors,
3. check if there was a Pythagorean triangle with a short side corresponding to each,
4. construct the composite triangle,
5. feed the values into equation (D) and see if an integral r came out.

But that would be to work backwards. What Alan did was use equation (D) in the form (E) to write three inequalities, so that, for each r, the computer need only run through a small set of numbers to check against (D). In my terms (and you should check these for yourselves), if \( k \leq l \leq m \), we have:

\[
\frac{3}{k^2} + \frac{1}{kl} + \frac{1}{lm} = \frac{1}{r^2} \quad \text{(E)}
\]

so \( k \sqrt{3r} \)

Since \( lm \) and \( mk \) \( kl \), \( \frac{1}{kl} < \frac{1}{r^2} \), so \( l \frac{3r^2}{k} \)

From (E), since \( \frac{1}{lm} + \frac{1}{mk} > 0 \), \( \frac{1}{kl} < \frac{1}{r^2} \) \( kl > r^2 \) \( kl \) \( r^2 + 1 \) \( kl \) \( r^2 \) \( 1 \)

\[
m = \frac{r^2(k+l)}{kl} \quad r^2(k+l),
\]

so \( m \ r^2(k+l). \)

Dear Paul,

10.48, 17.12.13

Just one more point ... having run the program one ought to check that the numerical possibilities are indeed realisable geometrically. ...

We’d reached the point, then, at which Geoff and I had an explicitly geometrical procedure for generating integer triangles and their in-circles – but no program, Alan had such a program – but no assurance that the numbers the computer threw out had geometrical correlates.
Dear Paul,  

19.34, 17.12.13

A hurried note as we are just on our way ‘out’ this evening. While having a shower I had the following idea ...

(Readers now need to rewrite (E) using the angles marked on the earlier figure.)

It seems to me that as long as \( a, b \) and \( g \) are in \([0, \pi/2]\), the condition

\[
\tan a + \tan b + \tan g = 1 \text{ EQUIVALENT to } \frac{1}{2} \frac{1}{2} \text{ ... Now solve the equation via the computer algorithm. With any solution we should be able to trace the variables back and so define angles } a, b, \text{ and } g \text{ whose sum is } \frac{\pi}{2}. \text{ We then construct six right-angled triangles with angles } a, a, b, b, g, g \text{ and they should fit together like a jigsaw to give us the existence of a triangle with the given (computer generated) sides ...}

Dear Alan,  

19.35, 17.12.13

What a neat way to get back to the geometry!

Dear Paul,  

00.19, 18.12.13

We had a good night out – with a further thought!

We should look at the well known formula for the parametric form of Pythagorean triples and find out what the collection of in-radii are. ...

Dear Paul and Geoff,  

13.12, 18.12.13

Attached is a revised and updated pdf file. Briefly, it shows that every right-angled integer triangle has a positive integral in-radius, and that, conversely, every positive integer is the in-radius of an integral right-angled triangle.

It also gives a reasonably efficient algorithm for finding all integer-sided triangles with a given integral in-radius.

Alan’s file is short (3 pp.) but brings together all the above. We have submitted it to The Mathematical Gazette, but I shall be happy to copy it to any interested reader.)

Dear Alan and Geoff,  

19.22, 16.1.14

Since our integer triangle correspondence I’ve been thinking about analogous quadrilaterals ...

Watch this space for a further triologue – or, if a reader of the foregoing joins in, tetralogue.

But the triangle correspondence took a further turn.

In connection with a piece in the Mathematical Gazette on a quite different topic, I brought Bob Burn’s attention to a student problem in the same issue which I had devised in the course of the above work.
In his reply of 21.3.14 he sent me the article you have before you in Mathematics Teaching. Like Alan’s piece, it had been through many drafts. Both Alan and Bob had arrived at the formula at the end of Section 2, which Bob had gone on to convert into the computer program forming Section 3. Our parallel universes had come together!

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