Further Pure Mathematics with Technology (FPT) is a new optional A-level Further Mathematics unit developed by Mathematics in Education and Industry (MEI). It requires students to have access to technology, in the form of a graph-plotter, spreadsheet, programming language and computer algebra system (CAS) for the teaching, learning and assessment. This article describes the development of the unit, including the rationale for the design decisions, and the implications for future developments of this type.

The development process

Although there have been a number of projects in England in the past 20-30 years to integrate the use of technology into teaching and learning in mathematics there are still many classrooms where technology is not used effectively. This issue was raised by Ofsted’s 2008 publication, *Understanding the Score:*

“Several years ago, inspection evidence showed that most pupils had some opportunities to use ICT as a tool to solve or explore mathematical problems. This is no longer the case ... despite technological advances, the potential of ICT to enhance the learning of mathematics is too rarely realised.”

The lack of realising this potential can be partly attributed to the lack of technology in the assessment of mathematics. In GCSE examinations students are allowed a scientific calculator, but not a graphical calculator, for some of the assessment. At A-level students are allowed a graphical calculator in all but one of their examinations; however, these examinations are designed to be graphical calculator neutral, that is - having a graphical calculator should offer no advantage to a student. It is not surprising that as the technology is not expected to offer an advantage in the examination that many teachers do not exploit its use for teaching and learning.

In addition to this there are no examinations where computer algebra systems (CAS) are allowed. The Joint Council for Qualifications requirements for conducting examinations (2012) state:

“Calculators must not... be designed or adapted to offer any of these facilities: -
• symbolic algebra manipulation;
• symbolic differentiation or integration;”

As a consequence of this there have not previously been any mathematics examinations in England that have the allowed the use of CAS. As a consequence of this CAS is rarely used in the teaching and learning of mathematics in English schools. This is missing an opportunity to take advantage of the benefits of using CAS. In *The Case for CAS* (2004) Böhm et al suggest that these benefits include making concepts easier to teach, supporting visualisations, saving time on routine calculations, and improving students’ perception of mathematics.

In 2008 MEI, in partnership with Texas Instruments, convened a seminar and invited leading experts to discuss ‘Computer Algebra Systems in the Mathematics Curriculum’. One of the main findings of this event was:

“An ICT-based qualification, where students have access to appropriate devices in the classroom and examinations would be useful. It could be a much more realistic qualification that allowed them to be better problem-solvers and mathematicians.”

In this context MEI wanted to drive the debate forward by exploring the possibility of having part of the A-level course involve the use of technology, including CAS, in a way that its use would be expected in the assessment, and consequently this would drive its use in the teaching and learning. The aim of this is to support the evolution of the role of technology in the mathematics curriculum from a computational tool, to a tool that allows for observation and conjecture as part of the mathematical process, as identified by Trouche (2004). MEI approached OCR, the examination board who administer the MEI A-level, who gave full support to the development of a new unit in the A-level Further Mathematics options.

The mathematical content of the unit

It was decided that the unit should be an optional...
unit for A-level Further Mathematics, and that the mathematical content of the unit would be taken from pure mathematics. This decision was taken because it was believed that a unit assessed by examination would fit best with the current specification, and that the study of pure mathematics offered more potential for examinations with technology than applied mathematics where technology is more useful for project-based continuous assessment. The unit will be taken by students who will be studying at least twelve units for A-level Mathematics and Further Mathematics. It is expected the unit will be studied alongside the Further Pure 2 unit.

It was a requirement that the topics in the unit contained mathematics that was not assessed elsewhere in A-level Mathematics, or Further Mathematics. As there are many optional units in A-level Mathematics and Further Mathematics, this limited the scope of topics for inclusion. The main criterion for inclusion was that topics had to be areas of pure mathematics that benefitted from students being able to use technology to access a large number of results quickly, and then to be able to analyse these results using their mathematical skills.

The topics chosen were:

- **Investigation of curves**
  A topic that can be greatly aided by access to a graph plotter, which allows students to plot a family of related curves using parameters, which can be changed dynamically. The use of CAS is also helpful in analysing the properties of the curves.

- **Functions of complex variables**
  Although students meet complex numbers in A-level Further Mathematics they do not consider functions of complex variables, such as \( \sin(z) \), or polynomials with complex coefficients. Using CAS allows students to investigate these, and this is complemented by the use of a spreadsheet for iterations using complex numbers.

- **Number Theory**
  Number theory is not studied at all in A-level Mathematics, or Further Mathematics; the use of a programming language, coupled with the access to CAS functions, makes this topic accessible to students at this level.

- The following example shows a curve where students are expected to be able to analyse the difference in the cases where; \( 0 < k < 1 \) and \( k = 1 \). The use of a *slider* makes this a realistic task and allows students to observe how its behaviour changes with the value of \( k \).

The full specification of the syllabus detailing the mathematical content can be found at http://www.mei.org.uk/fpt

**Technology**

The choice of the topics suggested that it was necessary for the students to have access to:

- Graph-plotter
- Spreadsheet
- Computer Algebra System
- Programming language

There are different pieces of software that feature these applications; however, as they will be used together, especially with CAS being used in all the topics, it is advantageous to use a piece of software that features all of these in a way that objects, such as functions, can be used across multiple applications.

There was a conflict when the software to use was considered. There were concerns about the requirement for a single, specific piece of software. An overt focus on a single specific piece of software could result in the unit being perceived as *training in the use of the software*, as opposed to *learning the mathematics*. In contrast to this is the concern that many teachers will be unfamiliar with some of the applications used, and will require teaching and learning materials that inform teachers about *how to use the software*, as well as covering the *mathematics*.
The compromise reached was that the specifications and features of the software allowed in the examination would be stated, but no specific software would be required: schools would be free to choose any piece of software that met the stated requirements. To support schools in making this choice teachers have been encouraged to seek advice from MEI about which software to use. TI-Nspire was chosen as the piece of software in which to write the teaching and learning resources, as it features all the required applications in a linked way, it is affordable for schools, and Texas Instruments have a strong record in supporting the use of technology in school mathematics. Schools are likely to be using other software in the teaching and learning, and they may choose to prepare students for the examination using other software packages in future years.

Assessment

It was felt that a unit assessed by examination would fit better in the current A-level specification. Consequently the assessment for this unit is an unseen, timed, written examination in line with the assessment of other units in A-level Mathematics and Further Mathematics. Students will have access to software in the examination, and the questions will typically start with an investigation into a problem using the technology with some follow-up questions in which students analyse their results mathematically. Students will submit a written response, but there is the possibility that, in future years, electronic submission of some, or all of their work, will be allowed.

3 (i) Create a program to find all the positive integer solutions to \(x^2 - y^2 = 1\), with \(x \leq 100, y \leq 100\). Write out your program in full and list the solutions it gives. [10 marks]

(ii) Show how the other solutions can be derived from the solution with the smallest \(x\)-value. Use each solution to give a rational approximation to \(\sqrt{3}\). [5 marks]

(iii) Edit your program so that it will find solutions to \(x^2 - ny^2 = 1\), where \(n\) is a positive integer. Write out the lines of your program that you have changed. Use the edited program to find a rational approximation to \(\sqrt{5}\) that is accurate to within 0.1%. [6 marks]

(iv) Explain why the edited program will not give any results if \(n\) is a square number. [2 marks]

Figure 2: Example examination question from the specimen paper

The regulators accepted the specification and the first examination is scheduled for June 2013. The expectation is that around fifty students from approximately ten schools will take the unit in the first year.

An example question from the specimen paper is shown in Figure 2 with a screenshot of what students would be expected to do on the software in Figure 3. The full specimen paper can be found at http://www.mei.org.uk/fpt

Implications of using CAS

One of the biggest changes to the assessment will be that students will have access to a computer algebra system in the examination. This changes both the types of questions that can be asked and the expectations of students in
terms of their responses to the questions. For example, on the specimen paper students are asked to find $f'(z)$ and show $f'(z) = 0$ has a repeated root for the function $f(z) = z^3 - (3 - 3i)z^2 - 6iz + 2 + i$.

![Figure 4: Using CAS to show that a function has a repeated root.](image)

In this example the second line shown, just solving $f'(z) = 0$, would be insufficient to gain full credit as only a single root to the quadratic is obtained; to gain full credit students would need to factorise and state that a repeated factor will result in a repeated root.

As students will have access to CAS they will be expected to select techniques and accurately present the outcomes; however, students will receive no credit for performing algebraic processes that can be performed by the software. This is analogous to the role of four-function calculators in other assessments at this level: in a question that requires the division of two large numbers students would be expected to select the process of division and accurately present the answer, but not be expected to perform a pencil and paper algorithm to obtain the quotient.

**Implications for future developments**

This unit has been developed to promote the effective use of technology in the teaching and learning of mathematics, and to provide concrete examples of the challenges and choices that need to be made when designing technology-based mathematics curricula and assessment.

Through developing the unit the following have been observed:

- **It is possible to develop technology-based mathematics courses within existing qualifications frameworks.**
- **The decisions made about the choice of topics demonstrate that technology is effective when it allows learners to access mathematical results quickly and efficiently so that students can make inferences and deductions based on these.**
- **When designing curricula and assessments that allow the use of CAS it is important that the emphasis is on learners selecting and using techniques, and not on the mechanics of implementing the techniques.**
- **Although different pieces of software have different qualities it is possible to identify the features that the software should have, instead of requiring a specific piece of software: for example, the graph-plotter should have the ability to control parameters with a slider or scroll-bar.**

All of these demonstrate that there are requirements/constraints when developing curricula, but that suitable innovation can produce interesting, exciting, and useful developments.

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**References**


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