The government inspired National Curriculum Review currently being carried out by DfE includes a report, ‘What we can learn from the English, mathematics and science curricula of high-performing jurisdictions?’ (DfE 2012), which makes for interesting reading. With regard to mathematics, it is evident that the focus on ‘jurisdictions where the content appears more challenging than that in England’ (p61) leads to the assumption, apparent within both the statements of ministers and also the Report by the Expert Panel (DfE 12/2011), that imitating the content, pace and pedagogy of Singapore and Hong Kong in particular, would enable us to improve the mathematical performance of English children. This assumption seems to have gained much ground amongst policy makers in recent months (Oates 2011), and despite the cautionary comments of the expert panel to the effect that it would be dangerous to indulge in ‘naïve extractions from transnational comparisons’ (p45), we seem to be facing a situation where the importation of Singaporean mathematics curriculum and teaching methods into an English National Curriculum is rather more probable than not.

Under these circumstances, it behoves all those involved in mathematics education to look very carefully at the evidence of both research and practice in attempting to ascertain whether such a strategy of drawing heavily upon curricula and methods from Singapore, or from Hong Kong and the Pacific Rim countries generally, is likely to succeed. There are many aspects of this ‘naïve extraction’, which cause alarm bells, some of which pertain to the very nature of our democracy and I have written about these elsewhere (Merttens 1/2012). Cultural differences between England and Singapore preclude any expectation that, in relation to mathematics, the focus on compliance and rote-learning could be transplanted without difficulty into many schools in the state sector of English education, where children both expect to enjoy rather than endure their classroom experience, and where conceptual understanding and creativity have long been strengths of the system. However In this paper, I focus on one specific and much lauded aspect of the textbooks used in Singaporean primary education – namely the heuristic of ‘Concrete – Pictorial – Abstract’ (CPA).

The above-mentioned CPA heuristic has been widely used to alleviate concerns about the importation of the Singaporean mathematics curriculum expressed by both maths educationalists and by teachers at the chalk face. There are many who fear that this curriculum, if implemented in English primary schools, would lead to an emphasis on rote-memorisation and procedural learning as opposed to the current and well-established accentuation of conceptual understanding and semiotic relation to daily practices. In simple terms, children in England expect to understand what they are doing in mathematics, and to make sense of their learning in relation to other areas of life, both real and imaginative. With a Singapore-style curriculum, these same children would be copying methods and rote-learning procedures without having necessarily grasped the underlying mathematical concepts and with little or no expectation that these procedures could be usefully related to other domains of experience. However, advocates of this approach assure us that the development of ‘understanding’ as well as procedural fluency is central to mathematics teaching in Singapore, and they point to the universally adopted CPA heuristic as evidence that this is the case.

Professor Lianghuo Fan, in his inaugural lecture, ‘Why do Singapore students excel in international mathematical comparisons?’ (2012) used the CPA heuristic to demonstrate one of the key ways in which Singaporean students are brought to a level of excellence not achieved by the majority in England. As the author of the textbook currently used in more than 80% of the country’s schools, he was well-placed to explicate this method of teaching. Showing a slide of two sets, one with 6 rabbits pictured and one with 7 rabbits...
picted, he demonstrated how the children could be shown a concrete representation of the addition $6 + 7$. He then showed a pictorial representation consisting of two sets, one with 6 large dots, and one with 7 large dots. Finally he moved on to reveal the abstract representation, $6 + 7 = 13$. I had previously had a chance to look at Singaporean primary maths textbooks at a seminar organised by Tony Gardiner at Clifton College in Bristol in 2010, when these were showcased as exemplar materials. On both occasions, I was struck by the fact that ‘concrete’ did not necessarily mean ‘concrete’ in the Piagetian sense of that word – i.e. as a set of manipulable objects. It could, within the bounds of the CPA heuristic, be pictures of objects on a page. I was also struck by the apparent lack of the mathematically derived models and images, such as number lines, number grids and so on, that we have considered important throughout the whole of my own career as a mathematics teacher. So what is the evidence that the CPA heuristic ‘works’? How do we know that if the ‘concrete’ part really is concrete and not another form of pictorial. The rabbits we were shown by Professor Fan and the pretty pictures of cars in the Singaporean textbooks, must actually be real rabbits and real cars, at least in the sense of being toys that children can handle and manipulate for themselves. Depicted objects are still representations of objects and so outside the scope of what Piaget meant by ‘concrete’ in this context. Piaget was unequivocal in his view that it was physical actions, which lead to learning – the general ability to move from sensori-motor to operational thinking depends on the internalisation of actions, namely that these actions on real objects are represented to oneself and hence internalised. So, for the heuristic to be theoretically defensible at all, ‘concrete’ must mean real life objects and not representations of these.

However, there are further, more intractable problems here and although these are problems at the level of theory, they are reinforced by an immense body of research. It was Margaret Donaldson and her colleagues (1978) who first brought the difficulties inherent in a Piagetian view of learning to the attention of the mathematics education community back in the mid-80s. In her influential book, Children’s Minds, she outlined how Piaget’s analysis misrepresented children’s thinking. She demonstrated that children were quite capable of what she termed ‘disembedded thought’ if the context in which the problem arose made human sense to them. Donaldson highlighted the importance of both the context in which a task arose and also the language in which it was phrased. Her work resonated with the writings of other researchers who performed experiments stimulated by those of Piaget and found that several factors proved crucial, particularly:
The 'Concrete – Pictorial – Abstract' Heuristic.

- The situation in which a particular task was embedded;
- The 'talk' by which it was presented and orchestrated;
- The presence of a helpful adult or more skilled participant.

As David Wood (1998) succinctly summarised the critiques, 'in relation to Piagetian tasks many researchers found that the child's ability to solve problems ... was influenced by factors such as task familiarity, the language used and the manner in which the experimenters introduced and explained the problem.'

If such research acted to undermine the naïve assumption that all children's learning proceeded through distinct stages, from sensori-motor to concrete operations to formal operations, the effect of this was also to pull the rug out from the matching pedagogy. This had been based on the desire to match Piagetian stages by moving children from concrete manipulation through representation in pictures to formal arithmetic, very much the correlate of the heuristic with which we are presented here. The work of Martin Hughes (1984) in Children and Number, showing that children as young as three were perfectly able to answer mathematical questions if they were presented within a context within which they made sense – such as 'what are two elephants and one more?' – provided a further blow to this theoretical position and its related teaching methods. Here were children clearly demonstrating that they could add and subtract in an imaginary context, which made perfect sense to them, with no concrete objects or even pictures available as support. The CPA heuristic begins to look dangerously simplistic at best, and seriously flawed if the Piagetian stages do not survive the critique.

Bruner (1966) produced what some argued was a more sophisticated version of this model when he outlined the distinction between three modes in which knowledge is represented: 'enactive', 'iconic' and 'symbolic'. However, Bruner was not asserting these as a progression in either teaching or learning, but rather as a means of analysing different ways in which knowledge is represented to and by children. Furthermore, although Bruner at this stage acknowledges a development of learning from practical to abstract contexts, he does not share Piaget's view that children have to reach a certain stage before they can perform certain types of task. Very young children can learn to perform and understand abstract calculations, according to Bruner, if they are given appropriate instruction or support and the context is one in which the task is meaningful and makes good sense to them.

It was the writings of Vygotsky, which accelerated the move away from stadial theories of learning. As his theories became more widely distributed and discussed amongst educationalists and researchers in Britain, the extent of their implication in relation to pedagogy and even curriculum content began to be appreciated. The idea that young children learn through a process whereby they internalise the orchestration of activity provided by a more skilled carer or tutor was to revolutionise both research and explanation in relation to teaching and learning. The influential Vygotskian thinker and writer, Wertsch, points out that with quite different types of task, the role of the adult is crucial in assisting the process described by Vygotsky (1978): the function exists first on the interspsychological plane (between people) and then, subsequently, on the intrapsychological plane ('within' the child). Wertsch comments, 'While forms of interspsychological functioning differ significantly, there seems to be at least one common tendency in how children come to master the situation definition of the task: they first participate in the execution of the goal-directed task on the interspsychological plane, and only subsequently do they recognise and master the strategic significance of their behaviours.' (1985, p166)

Post-Vygostky, there developed a clear understanding that Piaget's theory remains 'insensitive both to the importance of content and meaning in children's mathematical capability and to the central role played by the systems of signs and symbols made available to them by their culture.' (Wood, 1998) Piaget saw language as but one manifestation of the general symbolic function whereby the child comes to represent objects or events by means of signs. It follows that the recognition of the importance of both the formative and the pragmatic functions of language in relation to children's learning radically undermines his theoretical position. Therefore any assumption of a simple heuristic founded on the Piagetian stages of development is likely to be flawed. It is...
only when mathematical meaning is grounded in practical activities involving the use of models and images that the process of mapping symbolic meaning onto conceptual understanding is likely to occur. Successful teaching of mathematics will therefore necessitate the joint negotiation of meaning in relation to specifically mathematical contexts involving images, models and practical apparatus.

We arrive here at the root of our critique of the CPA heuristic. If children are to develop an understanding of abstract mathematical ideas and formal arithmetic then there need to be contexts in which a negotiated meaning and a joint understanding can be developed. Such contexts are supplied in English schools by a relatively well-developed and varied range of pedagogic tools, such as the beaded line, the number line, the 1-100 number grid, place value cards, the place value grid, Cuisinaire rods, Numicon, alternative number grids, etc. These images and models, whose use has been increasingly encouraged over the last decade, are crucial in that they provide the ground for increasing the importation of the CPA heuristic lock, stock and barrel is not advisable, two questions assume an urgent significance.

• What are the OTHER key factors in the mathematical success of Singaporean students?

• Which aspects of the curriculum can and should be imported to the English context?

Professor Fan addressed the first question in a comprehensive fashion in his inaugural lecture mentioned at the start of this paper. His analysis was thorough and did not suggest easy answers, but Singaporean teachers surveyed had highlighted the following factors: curriculum, textbooks, a teaching approach focussed on the tests, a great deal of test-practice, and good school-based schemes of work. Clearly, as Professor Fan also suggested, cultural features of the society play a crucial role. A highly motivated and compliant body of pupils, where the children of some of the lowest paid and most marginalised workers are not in school, helps the maintenance of quiet well-disciplined classrooms and schools with a competitive ethos. The presence of high-stake tests mean that most children are expected to have tuition and to engage in practice outside the school context to a degree unheard of in Britain, at least in the primary phase.
Memorisation and rote-practice are key components of education, not only in mathematics but throughout other subjects.

We have a distinctive ethos in mathematics teaching in England, where we want children to both understand and enjoy their learning. From the 1980’s onwards, we have recognised the limitations of a naïve adoption of unreconstructed Piagetian stages in relation to the development of mathematical thinking. Seymour Papert (1980) demonstrated how children can engage in solving mathematical problems in a ‘virtual’ environment and how these procedural skills could, under the right circumstances, be transferred to other contexts, and this was probably the final nail in the coffin of adherence to an over-simplistic version of Piaget’s stages. From that date onwards, educationalists, researchers and above all teachers have been working to develop pedagogic tools, which help children to develop their conceptual understanding in mathematics. The models and images mentioned above can and sometimes (Ofsted 2011) do serve us well here. They first provide the precise and dedicated mathematical context in which joint collaborative activity between teacher and children can take place. Subsequently, they orchestrate the transition of these functions from the interpsychological to the intrapsychological plane, enabling the child to orchestrate their performance using these same models when working alone. Finally the child will, with the collaboration of peers and teachers, structure the mathematical task using internalised versions of the image as a mental model.

At the same time, it should not be thought that I am suggesting that mathematics education is just fine in England, or in Britain as a whole, nor that there are not important things to learn from the Singaporean context. The focus on memorisation and routinised practice is one, which many agree we could, advantageously, imitate in English primary schools. For reasons, which have their roots in cultural shifts and in the expansion of technology, children should be encouraged to learn things by heart as soon as they enter nurseries and reception classes. As they are not likely to receive this training of the memory as a routine aspect of the daily rituals outside school. Further, as I have argued elsewhere, there are robust theoretical and academic arguments for the provision by teachers of a great deal more practice and a focus upon learning by heart. Jacques Ranciere (1991) mounts a powerful argument that, encouraging our pupils to learn by heart, to really pay attention to what they are learning and to engage in sustained repetition and practice, is not only a respectable pedagogical strategy post-Piaget, but also makes an assumption of equality – namely that all pupils are capable of producing this type of attention and memorisation. This will, in turn, enable them to advance their conceptual understanding in collaboration with the teacher. For Ranciere, equality of performance is therefore a presupposition rather than a goal – an interesting standpoint in relation to the Report of the Expert Panel.

We started with the report, ‘What we can learn from the English, mathematics and science curricula of high-performing jurisdictions’, and investigated the CPA heuristic which is said to underpin the Singaporean primary textbooks. The heuristic itself does not stand the test of rigorous scrutiny in relation to theory or research. Furthermore, it is difficult to find any evidence that it is a key factor in promoting conceptual understanding amongst Singaporean pupils, as is claimed. I have argued that, for both these reasons, it is unlikely to be a key factor in the success of Singaporean mathematics pupils in the international tests. I have also reasoned that its importation into materials used in English schools would be risky and, according to the best evidence available, extremely unlikely to bring about those improvements that we all wish to see. However, this does not imply that all aspects of the Singaporean curriculum are similarly suspect, and it is my belief that we could certainly emulate some aspects to our own advantage. It will be an interesting conversation between policy makers, teachers and researchers to see which ones these prove to be.

Ruth Merttens is Director of Hamilton Math Project and Educational Director of Hamilton Trust

References
ACME (2011) *Report on Primary Arithmetic* Published by Advisory Committee on Mathematics Education
THE 'CONCRETE – PICTORIAL – ABSTRACT' HEURISTIC.


Cowan, R. (2011) The development and importance of proficiency in basic calculation, Report from The Department of Psychology and Human Development, London, Institute of Education


Fan, Lianghuo (2012) Why do Singapore Students excel in International Mathematical Comparisons? at University of Southampton, Education School


Multi-digit subtraction in Learning and Instruction 19, 1-12


Ofsted (2011) Good Practice in Primary Mathematics: Evidence from 20 successful schools Ofsted


Torbejns, J. et al (2009) Efficiency and flexibility of indirect addition in the domain of multi-digit subtraction in Learning and Instruction 19, 1-12


WHERE’S THE MATHS?

For more images go to: www.atm.org.uk/resources/wheres-the-maths.html