EQUATIONS WITH TECHNOLOGY:
DIFFERENT TOOLS, DIFFERENT VIEWS
Paul Drijvers and Bärbel Barzel consider the potential of such tools for the students and the teacher

Nowadays, many digital tools are available to support the teaching and learning of mathematics. Teachers have a myriad of options to include technology in their teaching. However, different obstacles may prevent them from doing so. Beside practical issues, such as limited student access to technology, and limited teacher time for extended course preparation, pedagogical questions play an important role. What do the students really learn while using technology? Does the mathematical topic not become too trivial when there is a tool for doing the work? How does using technology affect students’ skill mastery and mathematical understanding?

These are important questions. Overall, recent insights seem to converge to the point that students need mathematical insight and skills to make appropriate use of technological tools. In fact, many studies suggest that there is an important, and subtle, relationship between students’ mathematical thinking and the opportunities and constraints digital tools offer. The task for the teacher, then, is to choose appropriate tools and to find corresponding tasks that guide the students to the targeted knowledge and skills.

This is a challenging task. As an example, in this article we investigate the relationship between the use of different digital tools and mathematical skills and understanding for the case of mathematical equations, an important topic in many mathematics curricula. The main goal is to describe how the different tools can be used while teaching and learning how to solve equations, and in particular how the use of these tools shapes the student’s skills and thinking. Of course, we cannot be exhaustive, and will therefore limit ourselves to the following tools: applets for solving equations, graphing tools such as graphing calculators, and computer algebra tools. For each of these tools, we will investigate its potential use for solving equations, and try to identify conceptual aspects involved in this type of tool use. This way, we address the following central questions: what digital tools are available for teaching the notion of equation in secondary mathematics education, and how do these tools and the corresponding tasks affect the students’ skills and understanding?

**Applets for solving equations**

Applets are applications that can be accessed through an Internet browser, and therefore do not need additional software installation at school. Usually, applets offer short interactive activities. We consider three examples related to equation solving, one applet visually supporting the balance model, another stressing the stepwise strategy, and a third one considering global substitution.

The first example is called *Algebra Balance Scales* and is taken from the National Library of Virtual Manipulatives, www.mattimath.com. As the name indicates, it uses the balance model, which can certainly help students to understand the notion of equation (Vlassis, 2002). Figure 1 shows how the applet represents a linear equation in terms of the balance model. It provides a virtual balance beam, with which the student can set up a given equation through the dragging and dropping of variables and units - Fig. 1 screen 1. Next, the student has to indicate operations, such as ‘+3’ or ‘-2x’, which will be carried out by the applet in the balance model - Fig. 1 screen 2.
If we consider the relationship between the use of this applet and the students' concept image, the applet clearly offers a manipulative view on solving equations as moving variables and units. Also, it stresses the notion of algebraic equivalence, visualized by the equilibrium of the balance. The applet promotes a view on solving an equation as manipulating a physical equilibrium through the same operation on both sides of the scale. Concerning the development of skills, the applet requires the student to decide on the next step to perform, but carries it out automatically. Therefore, the focus of the work is rather on the development of strategic skills than on paper-and-pencil procedural skills.

A limitation of this applet is that it only works for linear equations with integer solutions. Also, one may wonder about the position of activities with this applet in the learning trajectory: do students need to manipulate with physical blocks and balances before using the applet or does this interfere with the abstraction process? Do students automatically see the relationships between the balance and the corresponding algebraic representation? And, finally, how do students deal with the difference between $x$ represented as a cube symbol, and $-x$ as a balloon, even if $x$ is one single mathematical object, possibly with a negative value with the consequence of $-x$ being positive? How does this affect the students' conceptual development?

In short, this applet seems appropriate for the initial teaching of linear equations, in which the balance as a mental model of equivalence and the choice of appropriate solving strategy steps are the main learning goals. While using the applet, its limitations should be taken into account.

The second exemplary applet is called Solving Equations with Balance, and is taken from the Freudenthal Institute's website www.fi.uu.nl/wisweb/en/. Figure 2 shows how a student can use the applet to solve an equation. The student must be explicit about the next step to be carried out. The applet contains different versions, or levels, that differ in the amount of support that is provided while solving the equations. At the basic level, the student just needs to indicate the operation that is needed, and the applet carries out the algebraic calculation. At the next level, the student has to carry out the algebraic operations himself, but will get feedback on the correctness of the work. At the basic level, the student just needs to indicate the operation that is needed, and the applet carries out the algebraic calculation. At the next level, the student has to carry out the algebraic operations himself, but will get feedback on the correctness of the work. The third level is a self-assessment, which is corrected and graded by the applet. The fourth and final level is the test, which the teacher does not need to correct either. All together, the applet offers exercises, provides different types of feedback, and motivates through its game-like reward structure.

If we consider the relationship between the use of this applet and the students' concept image, the applet clearly offers a manipulative view on solving equations as moving variables and units. Also, it stresses the notion of algebraic equivalence, visualized by the equilibrium of the balance. The applet promotes a view on solving an equation as manipulating a physical equilibrium through the same operation on both sides of the scale. Concerning the development of skills, the applet requires the student to decide on the next step to perform, but carries it out automatically. Therefore, the focus of the work is rather on the development of strategic skills than on paper-and-pencil procedural skills.

A limitation of this applet is that it only works for linear equations with integer solutions. Also, one may wonder about the position of activities with this applet in the learning trajectory: do students need to manipulate with physical blocks and balances before using the applet or does this interfere with the abstraction process? Do students automatically see the relationships between the balance and the corresponding algebraic representation? And, finally, how do students deal with the difference between $x$ represented as a cube symbol, and $-x$ as a balloon, even if $x$ is one single mathematical object, possibly with a negative value with the consequence of $-x$ being positive? How does this affect the students' conceptual development?

In short, this applet seems appropriate for the initial teaching of linear equations, in which the balance as a mental model of equivalence and the choice of appropriate solving strategy steps are the main learning goals. While using the applet, its limitations should be taken into account.

The second exemplary applet is called Solving Equations with Balance, and is taken from the Freudenthal Institute's website www.fi.uu.nl/wisweb/en/. Figure 2 shows how a student can use the applet to solve an equation. The student must be explicit about the next step to be carried out. The applet contains different versions, or levels, that differ in the amount of support that is provided while solving the equations. At the basic level, the student just needs to indicate the operation that is needed, and the applet carries out the algebraic calculation. At the next level, the student has to carry out the algebraic operations himself, but will get feedback on the correctness of the work. The third level is a self-assessment, which is corrected and graded by the applet. The fourth and final level is the test, which the teacher does not need to correct either. All together, the applet offers exercises, provides different types of feedback, and motivates through its game-like reward structure.
If we consider this applet in terms of its potential for concept and skill development, we notice that, compared to the previous applet, the balance model is still the main frame of reference, but is not visually present. This avoids the risk of a cognitive overload from too many symbols, which all have to be understood.

The task is to simplify or reduce an equation to the form \( x = \ldots \), while maintaining the equilibrium of both sides of the equation, but the balance model is no longer visualized, even if some versions of the applet offer balance pop ups. Figure 3 shows how a student solves the equation \( 2x + 8 = -4x - 16 \) with paper and pencil. The writing clearly reveals the transfer of notation from the applet environment to paper-and-pencil and through its double arrows suggests a balance view. It is interesting to see how the student himself added the arrows on the left. This suggests the applet might also include arrows on both sides of the equations.

As the students, depending on the applet level, carry out the algebraic manipulations themselves, this applet focuses more than the previous one on the development of procedural skills. Also, its solutions are not necessarily integers, and as such the applet is mathematically richer than the one described previously. If, however, students do not manage to solve an equation, and would like to fall back to the balance model in a visualisation such as that shown in Figure 1, the applet is limited. A richer version would provide a button, or pop-up screen, that helps the students to re-construct the stepwise solving strategy (Yeo Shu Mei et al., 2008).

In short, this applet seems appropriate for combining the reinforcement of the balance model for equivalence with the practicing of procedural skills and the choice of appropriate solving strategy steps for solving linear equations.

The third exemplary applet takes a different stance. It is called Solving Equations with Cover Up Strategy, and is again taken from the Freudenthal Institute’s website www.fi.uu.nl/wisweb/en/. Figure 4 - screen 1 - shows a somewhat complicated equation. As soon as the student highlights the fraction in the first line with the mouse, the applet displays this term on the second line, and invites the user to enter its value. After entering the value 3, this is checked by the applet, and the procedure can be continued by covering up the square root part in the denominator in the second line - Fig. 4, screen 2. This leads to the solution.

Figure 4
Solving an equation with the cover up method

If we consider the potential of this applet for concept and skill development, we notice that the balance model is not so prominent anymore. Even if it still plays a role in the background, the main conceptual aspect here is the idea that we can cover up parts of an equation, even if they are as complex as the one encountered here, and see the equation as \( 8 - \ldots = 5 \). Instead of the dots we need 3, so this allows us to make a first step. This method, which we might call a global substitution approach, invites a global view of the equation and the expressions involved, and the recognition of the relevant part to cover up. The skill to identify an appropriate part of the expression is developed, and we believe this is not only important for solving equations, but also for acquiring a global view in algebraic
expressions in general. A limitation of the applet is that it is not possible to cover up an expression that appears in different places in the equation, which would prepare the user for solving equations such as \( \cos^2 x + 2 \cos x + 1 = 0 \).

In short, this applet seems appropriate for working on more complex equations involving parts that can be covered up, and to practice a global view of algebraic expressions and equations.

**Graphing calculator for solving equations**

Digital tools for graphing functions are widespread: there are applets for graphing, spreadsheet software offers graphing options, specific educational software is available for computers and for mobile technology, and dedicated handheld graphing calculators are on the market. In this section we focus on the latter, but meanwhile we would like to stress that there are no intrinsic pedagogical, or mathematical, differences between the different kinds of graphing tools. The students themselves can purchase graphing calculators, which guarantees permanent access independent from school facilities. Compared to other graphing tools, the screen size and resolution are limited.

Graphing tools offer a means to solve equations graphically by finding intersection points. Figure 5 shows such a procedure for solving the equation 
\[
\frac{x^2 + 100}{2x + 20} = 4.5
\]
on a Texas Instruments calculator. This equation emerged from a context problem in a Dutch national examination, during which students use their graphing calculators.

The first screen shows how the student enters the left-hand side of the equation as function Y1, and the right-hand side as Y2. In the next screen, the two graphs are drawn on an appropriate viewing window. In the third screen, the intersect procedure is started by selecting the two graphs and entering the procedure’s starting value. The fourth screen, finally, shows the numerical approximation of the smallest solution. Eventually, this method can be complemented by zooming in at intersection points, or by generating tables of function values. This procedure works for all types of equations in one variable, as long as an intersection point is visible in the viewing window. A limitation is that this technique leads to just one, approximated solution, and additional reasoning or re-applying the procedure is needed to find other solutions, or to be sure that all solutions have been found.

Even if these procedures may be slightly different for different types of graphing calculators, they share some overall characteristics. What are these characteristics, and how do they affect student skills and understanding? An important observation is that this procedure changes the view on equations. Solving an equation is no longer simplifying a balance, but comes down to the intersection of two graphs. A solution, therefore, corresponds to an intersection point, and is a numerical, approximated value. Also, the procedure leads to one, single solution and needs to be repeated for equations with multiple solutions. All together, the notion of algebraic equivalence, visualised by the balance model, is replaced, or complemented by, the functional view of intersecting graphs and, approximated coordinates of the points of intersection. As far as skills are concerned, the algebraic skills needed for simplifying equations are replaced by more technical skills, such as being able to choose an appropriate viewing window and to consider the option of other points of intersection outside the current view.

In short, graphing tools such as graphing calculators may stress a graphical view on solving equations as intersecting graphs, and a numerical view on solutions. For practicing algebraic skills, for acquiring exact solutions and for thinking about sets of multiple solutions, the tool seems less appropriate.

**Computer algebra for solving equations**

Computer algebra is software that allows for symbolic manipulation of algebraic expressions. This includes algebraic simplification, exact
differentiation and integration, and, of course, algebraically solving equations. Computer algebra systems (CAS) are available for both desktop computers and mobile platforms. Although originally not designed for educational purpose, CAS nowadays offers interesting opportunities for algebra education.

In many CASs it is possible to enter an equation and carry out the same operation on both sides. Figure 6 shows such a solution procedure using Geogebra 4.2, which comes down to a similar approach as the one described in the second applet - see Figure 2, even if the CAS interface is slightly more complex.

As is the case for the applets, the immediate feedback helps the student to see the effect of the algebraic transformation and to experience the efficiency of the solving strategy.

Figure 6 Stepwise-approach in a CAS environment

More advanced use of CAS involves using its built-in \texttt{solve} command. Figure 7 shows the solving of two equations using a computer algebra software tool, in this case TI Nspire. The first equation on screen 1 is the same as the one solved in the previous section, but this time one just enters the \texttt{solve} command. Two solutions are provided, both in terms of exact radicals. In screen 1, the approximate values of these solutions are given. It is remarkable, by the way, that the first solution is different in the fourth decimal from the one found in Figure 5. Figure 7’s screen 2 shows an algebraic solution of a parametric equation. The solutions are no longer numeric, but algebraic expressions. The second equation in screen 2 shows that the result is quite different if one changes the unknown from \(x\) to \(b\). Clearly, the CAS algebraic options are well beyond the capacities of the previous tools. As the user has no insight into how the CAS finds its results and the underlying - often quite sophisticated - procedures, using CAS may have a black-box characteristic.

Concerning the conceptual view on equations and the skills needed, CAS use as presented in Figure 7 is different from the stepwise approach in Figures 2 and 6. Rather than a balance view, an equivalence view, or a functional/graphical view, using the CAS \texttt{Solve} command in a somewhat black box way as suggested here appeals to a more abstract, symbolic view, and invites other types of conceptual development. First, a student needs to be able to read the outcomes. For example, the first solution in screen 1, \(\frac{-1(41-9)}{2}\), is represented in a way that may be uncommon to students. The ability to read through algebraic expressions may be developed through tasks such as the global substitution activities presented in the applet section. Second, students need to be aware of the difference between exact, and approximated, solutions. A third, more technical notion concerns the difference between \(b \cdot x\) and \(b \cdot x\), the first representing a word variable, and the latter the product of two literal variables. Fourth, the two equations in the right-hand screen appeal to an awareness of the role of the unknown, in particular in equations, which contain more than one variable, solving with respect to \(x\), comes down to expressing \(x\) in terms of \(b\), or isolating \(x\).
This means that the students’ conceptual image of solving equations needs to be extended: solving also can mean expressing one variable in term of others, and solutions in this case are no longer numbers, but can be algebraic expressions! The latter is sometimes difficult for students, particularly if they see algebraic expressions and equations as not real answers, but as you don’t know which number they stand for. In the literature, this is referred to as the lack-of-closure obstacle.

In short, computer algebra is a rich and sophisticated tool, which can be used appropriately for both stepwise solution strategies in an early phase tool, as well as for expanding the students’ conception of solving equations, and for inviting a global view on algebraic expressions in a later phase. Also, its use can be productive for combining different views on solving equations through the use of different representations. Students can perceive equivalence simultaneously in different ways: in a table - finding same values, in a graph - finding points of intersection, and in an algebra window through balancing two expressions.

**Conclusion**

The examples above show that the type of digital tool, of course coordinated with the type of tasks and activities, highlights specific conceptual aspects of solving equations, such as a balance view, an equivalence view, a global substitution view, a functional view, or a symbolic view. As such, using the tool affects both students’ concept image and skill mastery. Meanwhile, the tool may ignore other, possibly important aspects of the notion of solving equations. The pedagogical task for the teacher, therefore, is to decide whether the use of a tool, with its specific tasks and techniques, invites the targeted conceptual development.

How can teachers take such decisions? In general terms, alongside factors such as the tool’s availability and ease of use, the following questions may guide the process of considering the use of a digital tool.

- **Do the activities with the digital tool invite the desired concept image?**

  As we have described above, different tools and different activities invite different concept images. Digital tools shape the students’ thinking; in the same way that the students’ thinking shapes the way the tool is used (Hoyles & Noss, 2003). This idea is elaborated in instrumentation theory, which highlights the close relationship between techniques for using digital tools and concept images or schemes (Trouche & Drijvers, 2010). It is important to be aware of possible tool-specific add-ons to mathematical concept images that may lead to cognitive overload, or misconceptions.

- **Is the concept image, as it is reinforced by the tool use sustainable, or does is show immediate or future limitations?**

  It is important that the concept image as it is evoked by the use of the digital tool is sustainable, so that it can be used in future. Also, one needs to be aware of the constraints and limitations of the digital tools. For example, the balance view on equations is more sustainable than the global substitution view, and that the applet shown in Figure 1 only works for integer solutions, is a limitation that can be overcome later. The different representations of $x$ and $-x$, however, may be a less sustainable aspect of the concept image, which has to be dealt with.

- **Are the techniques for using the digital tool in line with the targeted paper-and-pencil skills?**

  The ways in which a digital tool is used and the techniques it invites are closely related to the development of students’ understanding, but also to their skill development (Kieran & Drijvers, 2006). Therefore, one has to be sure that the digital techniques, and the targeted paper-and-pencil techniques, can be reconciled in the students’ minds.

As we mentioned before, to reflect on these questions, and to decide on a pedagogically sound integration of digital tools, is the task of the teacher.