The Cardioid film can be viewed here: http://www.atm.org.uk/free-resources/

The Cardioid. A Mathematical Film

Produced by Sir John Cass College

Designed and directed by Trevor Fletcher

Photography and technical advice by Polytechnic Films Ltd.

15 minutes. Silent.

This film illustrates the properties of the cardioid which are connected with its generation as a one-cusped epicycloid. These properties are of wide importance as, suitably interpreted, they are general properties of all epicycloids. It is, therefore, possible to comment on the film at many levels, and as it becomes increasingly familiar a viewer will wish- to focus his attention on different things on different occasions. A sound track "would be a disadvantage as it would always persuade the audience to look at the same things, and it is preferable for a teacher showing the film to give a commentary suited to his class.

These notes provide material for commentaries at two levels. The first part of the notes is suitable for, say, a Sixth Form with some knowledge of the simple properties of the cardioid, done perhaps by coordinate methods; while the second part is suitable for more advanced university audiences and gives the generalisations of all the theorems involved. These are stated without proof, and can be regarded as questions about the film. The reader may find some of these results difficult to prove, but the best method of proof is nearly always a pure one depending directly on the concepts illustrated in the film. Indeed, when the full implications of these moving diagrams are properly understood it will be found that many questions on epicycloids can be solved mentally when they would be quite difficult if approached by more static methods. The properties of hypocycloids are very similar, and some typical ones are illustrated in our earlier film 'The Simson Line'.

The film is in five parts separated from one another by short fades.

Part 1 - **The Double Generation of the Cardioid**

A circle is drawn in. For convenience its radius will be taken, as unity. It remains fixed throughout the film, and is called the pitch circle, Two circles of equal size appear and roll round it, (The reason for using two rolling circles instead of one becomes clear later). A point on each of the rolling circles traces in its locus, which is a cardioid, that is an epicycloid with one cusp. The chord joining these two points passes through the cusp of the cardioid, it is of constant length, and its mid-point moves round the pitch circle,

Next we see a second generation of the curve. The cardioid is also generated by rolling a circle of radius two pericyclically round the same pitch circle as before. Two points on this rolling circle generate the curve, and the cuspidal chord soon earlier is a diameter of it.

Finally the scene shows the two generations taking place simultaneously.

Part 2 - **The Tangents**
The two original rolling circles reappear, followed 'by two circles of twice the radius which roll round keeping in step with them. The function of the new circles is not clear until the generating points draw in a diameter on them, and then these diameters are seen to be the

tangents of the cardiod. The tangent to the curve is carried round by a circle twice the size of the circle which carries round the point on the curve.

The second generation of the curve gives rise to a similar configuration, and the circle rolling pericyclically with the tangents again has twice the radius of the circle involved in the point generation --so that its radius is four. Since two points on the circle radius two rolling pericyclically generate the curve two diameters of the circle radius four touch the curve.

At the end of this part of the film all the circles carrying tangents appear simultaneously.

Part 3 - **The Normals**
The rolling circles are rocked to and fro to draw attention to their points of contact with the pitch circle. These are the instantaneous centres of their notion. In each case the nomal to the cardioid passes through, the point of contact. The normals are drawn in blacks Among other results it can be seen that the normals at the ends of the chord through the cusp of the cardioid are perpendicular, and intersect on the pitch circle.

Part 4 - **The Evolute**

The evolute of a curve can be defined as the envelope of its nomals. The normals to the cardioid are drawn in exactly the sane manner as in the previous scene of the film, and their envelope then appears. It is another cardioid, one third of the size of the original one. The three to one ratio in size is emphasised by introducing the tangents to the first curve as well and drawing in the loci of the points of intersection of both tangents and normals. These loci are circles, and are parts of the orthoptic loci of the cardioids, (The orthoptic locus of a curve is the locus of points from which a pair of perpendicular tangents can be drawn. The circle radius three is only part of the orthoptic locus of the cardioid; the remaining part is a trochoid which does not appear in the film).

Part 5 - **Recapitulation**

The final part of the film recapitulates what has gone before, and also shows a number of combinations of elements which have not been seen previously. As a final flourish all the features which hare been seen earlier are superimposed.

Generalisation of the Properties

Part I

When a circle radius b rolls externally on a fixed circle radius a (where a and b are mutually prime integers) b points on it generate an epicycloid with a cusps. The second generation is provided by (a + b) points on a circle radius (a + b) rolling pericyclically on the same pitch circle. The cusps of the epicycloid generate a hypocyoloid with (a + b) cusps on the circle rolling pericyciically, (The case a = 2, b = 1 is partially illustrated at the very end of the film "The Simson Line"; the actual circles involved are not drawn in there, but we see the three-cusped hypocyoloids moving so that they always pass through the cusps of a two-cusped epicycloid while their own cusps move round it).

Part 2

The tangents to any epi- or hypocycloid can always be constructed as positions of a diameter of a rolling circle of twice the radius of the circle which provides the point generation of the curve.

A further generalisation is possible. If a rolling circle of k times the radius is used then a k-cusped hypocyoloid inscribed in it envelops the stationary curve.

Part 3

The construction of the normal as shown applies in the general case.

Part 4

The evolute of an epicycloids with n cusps is a similar curve with apses situated at the cusps of the original curve. The ratio of the linear dimensions of the two is n;(n+2), These facts lead to many corollaries. For example; from each point on the pitch circle (a + 2b) normals can be drawn to the curve, the feet of (a + b) of these are the vertices of a regular (a + b) - gon, and the feet of the remaining b are the vertices of a regular b – gon. The tangents at the (a + b) points concur at a point on the apsidal circle, and the tangents at the remaining b points concur at the diametrically opposite point of the apsidal circle.

Many more corollaries can be deduced from the results displayed here, and there is an almost unlimited number of drawing exercises available in illustrating the analogous properties for other curves of the family.

As a result of making this film I have come to realise how strikingly beautiful the epi- and hypocycloids are, and I am quite certain that they deserve more attention then they receive in most mathematics courses. I hope that this film will do something to increase their popularity.

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T. J. Fletcher

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