

The Fractal Geometry of Numbers

David Gibson

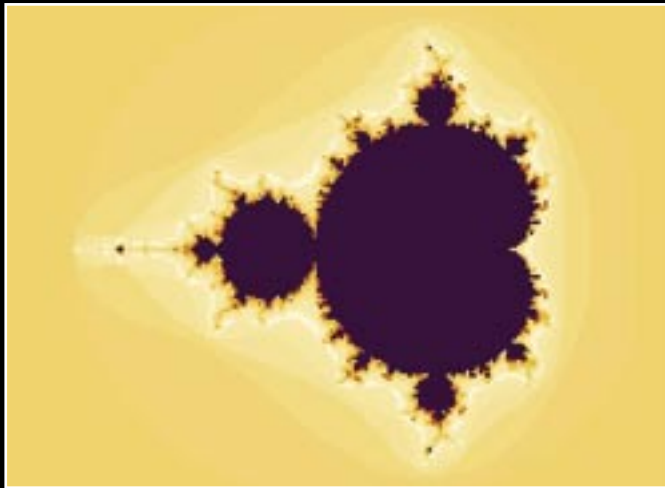


Figure 1a

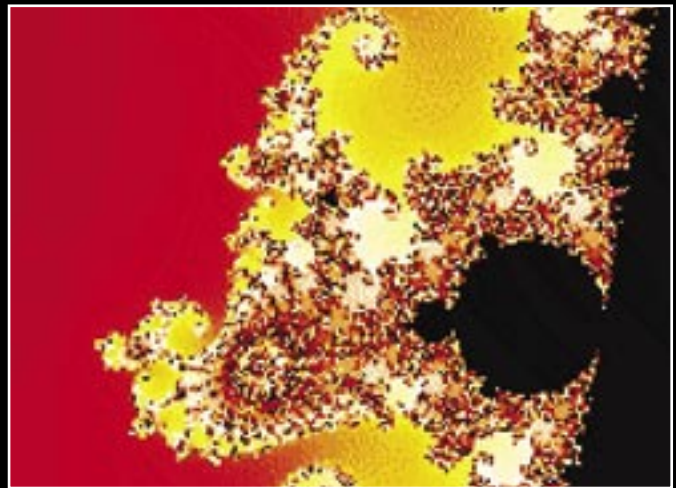


Figure 1b

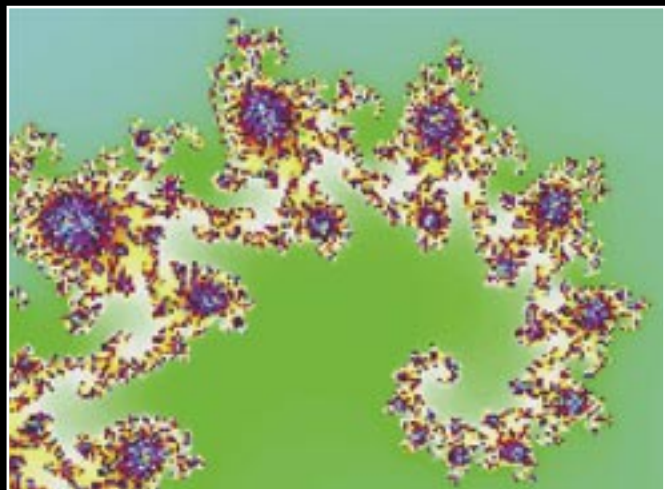


Figure 1c

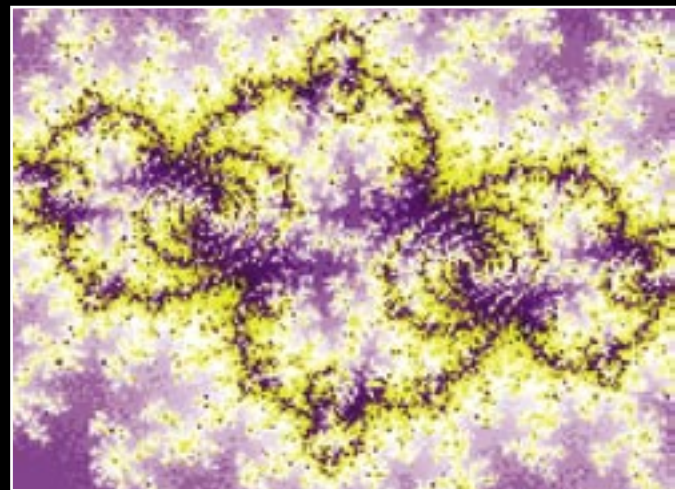


Figure 1d

In August 1994, Wired Magazine published an interview entitled the "The Geometric Dreams of Benoit Mandelbrot". Mandelbrot coined the term "fractal" in the 1970s and the stunning beauty of his dreams has created much common ground for mathematics, art and nature. Fractal images resonate deeply with the subconscious mind. Indeed the Mandelbrot Set (Fig 1a) has been called "The Thumbprint of God". It can be magnified without end revealing innumerable natural archetypes (Fig 1b-c).

The interview begins as follows:

Q: Your book is called "The Fractal Geometry of Nature". What is the fractal geometry of nature?

Mandelbrot: "The geometry of Nature is fractal to the extent that if you look at many shapes in nature — clouds, trees etc. small parts are the same as big parts; that's the definition of "fractal"."

Q: If we had no computers but an abacus, what would we know about fractals?

Mandelbrot: "Nothing"

In fact, there is more to the abacus than meets the eye — there are myriads of fractal patterns which have lain buried in its beads for the last five thousand years. Graphics are only a trigger for an infinite image — but the imagination can endlessly embroider the boundaries. The "Fractal Abacus" uses computer-generated images to make these patterns visible.

Adults are often surprised at the way children readily grasp the full significance of these complex, yet beautiful images of number. Children lack adult preconceptions. There is simplicity within this complexity, and a resonance between form and function which fires the imagination. It is a bridge between geometry and number, and between the aesthetic and the functional. The Fractal Abacus is more than a classroom resource — it is the gateway to another world. It helps to develop the conceptual framework which enables us to check our change at the shop, and at the same time presents opportunities to enlarge the imagination beyond the linear boundaries of blocks and rods. It is an opportunity (as William Blake might have put it) "To hold infinity in the palm of your hand."

We can represent Base Ten "entities" as shown in Fig 2 a fractal pattern is already beginning to form. This representation of "one" as a white ten pointed star is, however, a very great simplification

Numbers as fractals

One should be represented as ten tenths, or a



Figure 2

hundred hundredths, or a thousand thousandths etc. (Fig 3) On this basis it is important to remember that these

(and all of the graphic images which follow) are simplifications, since they have to stop somewhere, but the imagination can carry on embroidering them with



ONE AS A HUNDRE
D HUNDRETHS

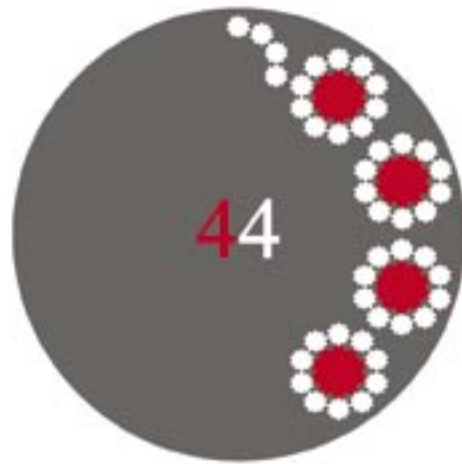
Figure 3

arbitrary colours.

When a number is not an exact power of ten it should, strictly speaking, be represented as a broken pattern (Fig 4).

In practice magnetic beads are used. A box ensures that no more than nine counters can be placed in a column (Fig 5). Circular Tens Frames are used to change ten Ones for a ten, or ten tens for a hundred, or vice-versa (Fig 6).

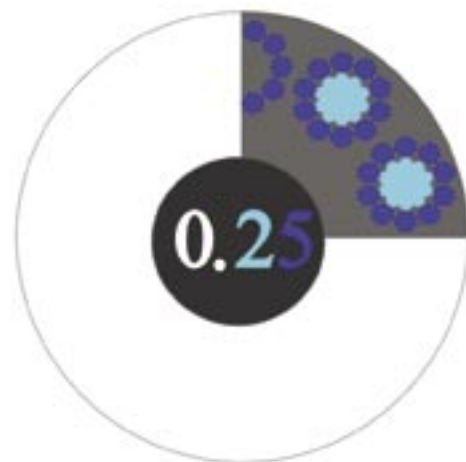
Many of us assume that God created our Base Ten



FORTY FOUR



A HALF



A QUARTER

Figure 4

number system (ie. hundreds, tens and units) in the Beginning. He didn't, He did, however, create humans with five digits on each hand. Without fractals,

Variations of Cantor's Comb and The Number Line fractal (Fig 8c), serve any number base. Base three, six and ten (so called triangular

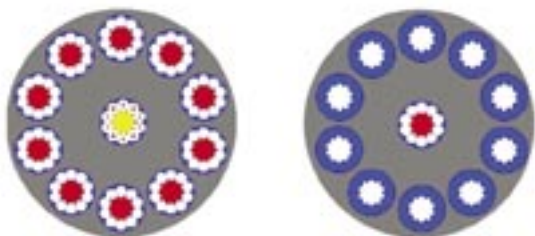
Figure 5



ABACUS BOX

No more than nine counters per column are allowed

Figure 6



CHANGING CIRCLES

The ten counters on the outside can be changed for the inside counter (or vice-versa)

Number bases

Earthlings struggle with other systems. The software version of the Fractal Abacus shows that a fractal mosaic of stars or polygons can (in principal) model any integer number base (Fig. 7).

Base Two is an exception & it can, however, be modelled by the fractal (Fig. 8a).

ONE



SIX



THIRTY SIX



TWO HUNDRED AND SIXTEEN

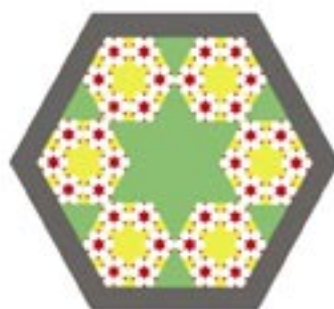


Figure 7

numbers \tilde{O}) can be represented by triangular fractals (Fig 9a,b,c).

In Base Three this \hat{O} triangular \tilde{O} fractal is identical to its \hat{O} polygon \tilde{O} form (a \hat{O} triangle \tilde{O} is a three-sided polygon). Each white triangle contains three thirds, and each of those contains three ninths etc. (Fig 10).

\hat{O} EIGHT \tilde{O} AS BASE TWO \hat{O} H \tilde{O} FRACTAL

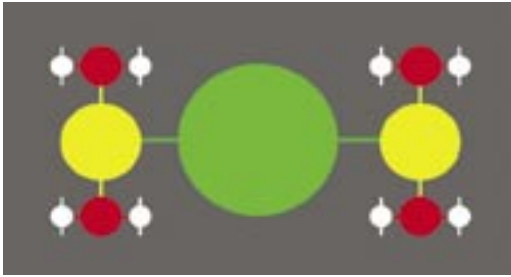


Figure 8a

\hat{O} EIGHT \tilde{O} AS CANTOR \tilde{O} S COMB IN BASE TWO

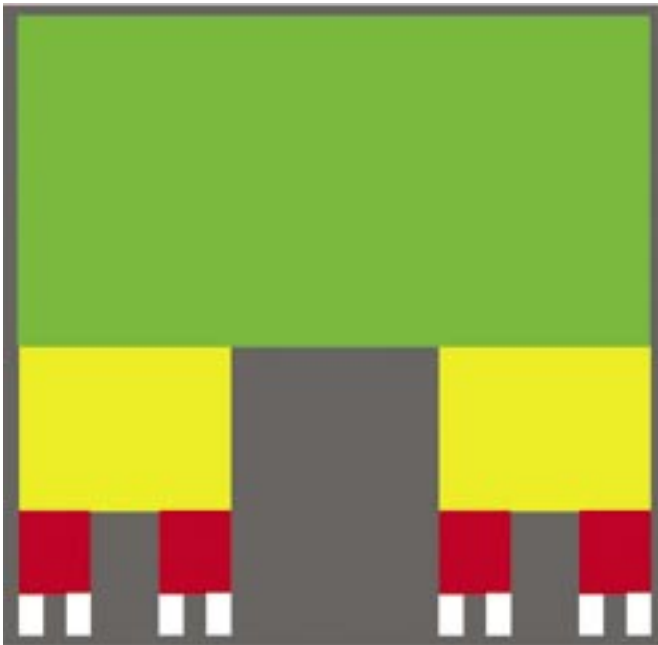


Figure 8b

\hat{O} EIGHT \tilde{O} AS BASE TWO \hat{O} NUMBER LINE \tilde{O} FRACTAL



Figure 8c

Notice that, if Fig 8b were a three dimensional solid representing the number \hat{O} eight \tilde{O} , then Fig 8c would be the view from underneath it \hat{D} in effect a colour-coded number line exhibiting infinite fractal properties.

One is two halves, four, quarters, eight, eighth and so on. The end result is a tower perched on an infinite number of \hat{O} bristles \tilde{O} of infinitesimal height and thickness!

BASE THREE



Figure 9a

BASE SIX

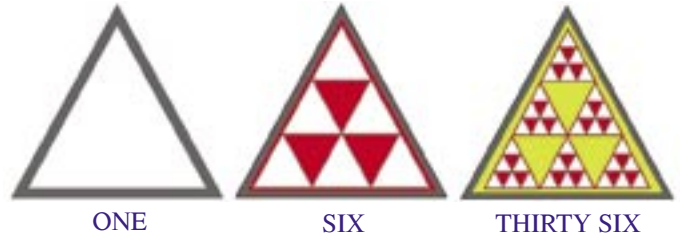


Figure 9b

BASE TEN

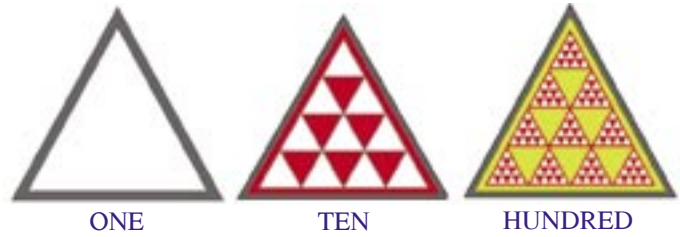


Figure 9c

It is immediately clear that a third is an exact ternary (Base Three) fraction (0.1). In Base Ten it is a recurring fraction (0.33333 \hat{E}).

\hat{O} Square \tilde{O} number bases (four, nine, sixteen etc.) can be

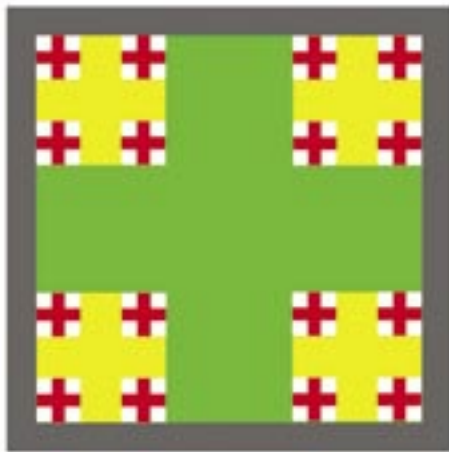
TERNARY FRACTIONS



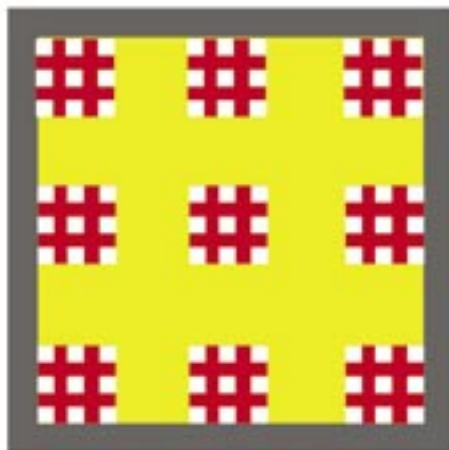
Figure 10

represented by square fractals (Fig 11). In Base Four the "square" form is identical to the "polygon" form D since a square is a four-sided polygon. More generally, "rectangular" number bases (six, eight, ten etc.) can be represented by rectangular fractals (Fig 12). In Base Twelve, 4×3 and 6×2 arrays are both feasible. The "Number Line" fractal (Fig 8c) is a more generally applicable "rectangular" form ($b \times 1$ D where b is any number base).

“SQUARE” BASES



SIXTY FOUR D BASE FOUR



EIGHTY ONE D BASE NINE

Figure 11

These rectangular fractals are reminiscent of the simple tens frame which is commonly found in the classroom. It is a piece of card divided into ten sections (Fig 13). Objects can be counted as they are placed on each of the sections until the card is full. A full card represents a "ten". Further counting requires a new start on an

BASE SIX



ONE



SIX



THIRTY SIX

Figure 12a

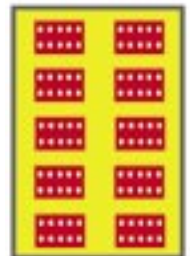
BASE TEN



ONE



TEN

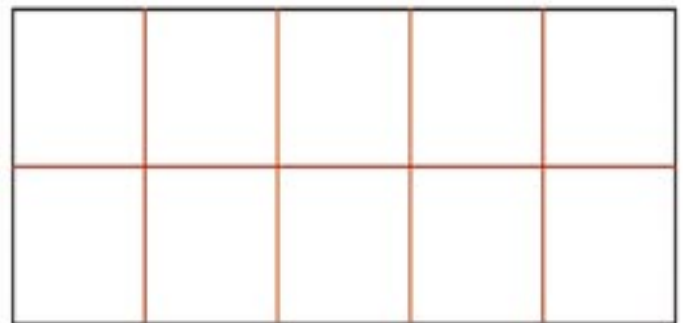


HUNDRED

Figure 12b

empty card. Ten cards can be placed on a frame ten times the size to make a "Hundreds Frame".

The software can be configured to simulate this process,



A TYPICAL “TENS FRAME”

Figure 13

creating a Fractal Abacus with a rectangular form. Each "ten" counter has ten white bars (two rows of five). Grouping numbers in fives is as old as counting itself (since we have two hands) D indeed this has been the norm in most traditional abacus designs. For children, five serves as a natural "datum" for number facts so $8 = 5 + 3$, $3 = 5 \text{ D} 2$ etc.

Nevertheless, the rectangular system is more unwieldy and less natural and aesthetically pleasing than the

ÔpolygonÕ form. As the North American Indians pointed out ÒNature works in circles, only the white man thinks in straight lines.Ó

The Fractal Abacus software can move seamlessly into negative numbers. A ÔnegativeÕ number is identical to its positive counterpart Ð but it has a different background colour. Two positive (or negative) counters will happily share the same box, but a positive and a negative counter will annihilate each other (rather like matter and anti-matter). Conversely, ÔzeroÕ can be thought of as a positive number (of arbitrary value) ÔstuckÕ to its negative counterpart.

This has some very interesting applications Ð it can, for instance, be used to develop new graphical subtraction techniques. It becomes very clear that subtraction is, in fact, nothing more than the addition of a negative number. Furthermore, multiplication / division involves the repeated addition of a positive / negative number.

The Fractal Abacus can, in principal, serve as a calculation tool for any real number operation in any number base using any conceivable method. It describes all aspects numbers and numeric processes geometrically Ð generally these processes are described symbolically. Consider how difficult it is to fully understand the number Ô44Õ. Firstly, Ô4Õ is an arbitrary symbol to represent four entities. Secondly, these symbols represent different entities depending on their relative positions Ð that is we have four ÔtensÕ and four ÔonesÕ. Already there is a very high level of abstraction. More often than not, by the time children start performing operations like addition, subtraction etc. they are losing the thread and rote learning is beginning to kick in to some degree. The mechanics of rote become so ingrained that adults are frequently unaware of how little they understand about number. We are left with a situation where numeracy might be defined as ÒThe ability to cope when a calculator breaks downÓ.

Mandelbrot blames Plato for this obsession with symbols and for Òthe anti-geometric, anti-picture binge in which so many scientists have indulged for so longÓ. He says ÒThe resulting abhorrence lasted until my work on the Mandelbrot SetÓ. ÒI had this accumulation of geometric ideas dying to burst out of my head into other peopleÕs eyes.Ó

SierpinskiÕs Sieve (Fig 14) is one of a number of fractal patterns which have been generated from number bases. Yet it would appear that, until now, nobody has turned the relationship around. Colour coding enables us to see that numbers can be described by fractal geometry. It hardly seems credible

that there could be an unopened door within simple day-to-day mathematics - but thereÕs still more to be discovered.

It would seem that this is a valuable discovery because it is beautiful and accessible. Children have few preconceptions, so the Fractal Geometry of Numbers is never lost on them. A geometric pattern with infinite colour can only exist in the imagination Ð and the Fractal Abacus is the key which unlocks that door Ð turning the mundane into the magical. It is rewarding to see that magic burst out of their eyes.

SIERPINSKIÕS SIEVE

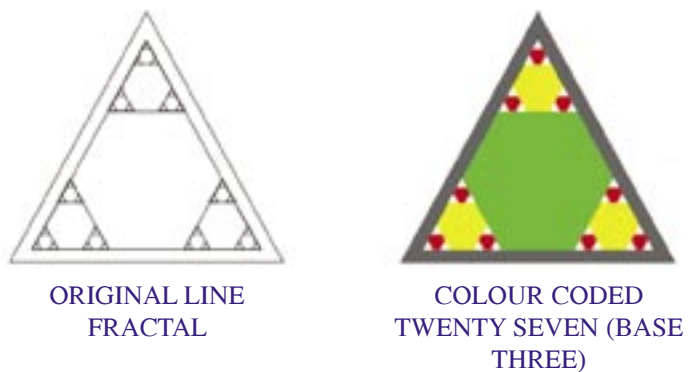


Figure 10

David Gibson is Director of Fractal Works Ltd. David is in the process of creating a website, www.fractalabacus.com, with downloadable software and other information about their work. It should be up and running by April 2002.

David Gibson developed a range of hardware, software, games and puzzles related to the Fractal Geometry of Numbers. He discovered the link between fractals and numbers in 1998.

Fractal Works Ltd., Cape Farewell, Collingwood, Golden Bay, New Zealand
Tel: +64 3 24 8680 Fax: +64 3 524 8680 E-mail: david@fractalabacus.com

Wired transcript: www.wirednews.com/wired/archive/2.08/mandelbrot.html
More on abacus design: www.ee.ryerson.ca:8080/~elf/abacus

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