

With a spreadsheet introducing gradient functions and differentiation

David Crawford

In this article I will describe the results of a short experiment I carried out with a group of top set Year 10 pupils, using spreadsheets to teach them some early ideas of calculus. In particular I was looking for an alternate method for teaching the idea of finding the gradient functions of various polynomial curves. I felt that introducing them to the ideas of calculus two years earlier than is normal would be an appropriate learning experience. This is because the mathematical concepts required (gradients of lines, limits of sequences and finding formulae from tables of values) were all ideas they had met and been successful with in the past.

One of the most appropriate technological packages for use within a mathematics classroom is a spreadsheet as the programming skills necessary to set up simple calculations are reduced to a minimum. This is particularly true when the same calculation is to be performed repeatedly. It was this property that I intended the pupils to use when establishing formulae for the gradients of various curves. However, I was aware that if "you sit a pupil in front of a blank spreadsheet for the first time then it is very unlikely that... progress will be made" (English 1993). This is because constructing even a very simple spreadsheet to perform a task requires a great deal of prior knowledge, like how to input formulae and how to perform multiple calculations and iterative procedures using fill commands. To get round this I first asked them to use some pre-

written spreadsheets so they could see how to operate them. Then we moved on to study how these particular spreadsheets worked by examining the processes involved in the calculations and the associated syntax used. Once they were familiar with the structure of spreadsheet programming, I felt they were ready to try and write some spreadsheets of their own.

Before describing my class's work I will first examine some of the reasons behind my choice of calculus as a topic to be taught using a spreadsheet. I wanted the topic to be relatively unfamiliar so that the pupils would have few preconceptions of the topic. This would enable me to compare how quickly these pupils assimilated the results with the classes of sixth form pupils, which I had taught in the past by a more rigorous "chalk and talk" approach. I also wanted a topic that was academically challenging but possessed some simple rules that could give success. Polynomial differentiation seemed to fit these criteria as well as offering a method that matched the structure of problems that spreadsheets are well designed to solve. It also allowed me to have a way of assessing whether the pupils had assimilated the ideas by asking them to predict what the gradient functions for different simple curves might be without having considered those particular curves on the computer.

Before starting to write spreadsheets, we spent a single classroom lesson recapping the gradient of

a straight line between two general points and considering the way they could try to find the gradient of a curve at a point. Then the class was set the task overnight of trying to design a suitable spreadsheet for the task. This turned out to be rather difficult for many of them as it seemed there was still some confusion about the way spreadsheets worked and indeed over the actual values required to calculate the gradient. But once in the computer room, and with a little prompting, all the pupils soon produced a spreadsheet, with a typical resulting spreadsheet giving a sequence of gradients of secants tending to the gradient of $y = x^2$ at $x = 1$ being shown in Figure 1. I asked the pupils to record their results on a record sheet (see Figure 2) to give them a structure to help them to find formulae for the gradient functions and to allow me to refer to them quickly. After a slow start finding the gradient function of $y = x^2$, the class soon got to grips with the process and started making rapid progress in a number of different directions.

A group of girls kept things simple in their alteration of the original function by just adding a

constant to the function $y = x^2$. (See Figure 3a for a description by Rose.) What made their approach interesting was that they did not just accept the computer generated results as automatically correct as most of the class had done. Instead they sought to justify what they had found by devising a 'proof', in this case a geometrical one, as they had been taught to adopt when carrying out investigations in previous years.

Another group quickly established a general formula for the gradient function of x^c and then went on to look at adding variable numbers of x 's to the functions whose gradients they had previously established and managed to obtain a general formula for this. (See Figure 3b for a discussion of their results by Olly.) Yet another group also established quickly the general formula for x^n and then went on to look at constant multiples of these functions giving a further formula. (See Figure 3c for an explanation by Peter.) Most of the rest of the class also managed to obtain the same initial general formulae by the end of the lesson and began working on extensions of their own.

x	Difference(h)	x+h	f(x)=x^2	f(x+h)=(x+h)^2	gradient	point x=?
1	4	5	1	25	6	1
1	2	3	1	9	4	
1	1	2	1	4	3	
1	0.5	1.5	1	2.25	2.5	
1	0.25	1.25	1	1.5625	2.25	
1	0.125	1.125	1	1.265625	2.125	
1	0.0625	1.0625	1	1.12890625	2.0625	
1	0.03125	1.03125	1	1.06347656	2.03125	
1	0.015625	1.015625	1	1.03149414	2.015625	
1	0.0078125	1.0078125	1	1.01568604	2.0078125	
1	0.00390625	1.00390625	1	1.00782776	2.00390625	
1	0.00195313	1.00195313	1	1.00391006	2.00195313	
1	0.00097656	1.00097656	1	1.00195408	2.00097656	
1	0.00048828	1.00048828	1	1.0009768	2.00048828	
1	0.00024414	1.00024414	1	1.00048834	2.00024414	
1	0.00012207	1.00012207	1	1.00024416	2.00012207	
1	6.1035E-05	1.00006104	1	1.00012207	2.00006104	
1	3.0518E-05	1.00003052	1	1.00006104	2.00003052	
1	1.5259E-05	1.00001526	1	1.00003052	2.00001526	
1	7.6294E-06	1.00000763	1	1.00001526	2.00000763	
1	3.8147E-06	1.00000381	1	1.00000763	2.00000381	
1	1.9073E-06	1.00000191	1	1.00000381	2.00000191	
1	9.54E-07	1.00000095	1	1.00000191	2.00000095	

The values in column F are tending to the gradient of the curve at $x=1$ ie 2

EQUATION $y =$		PREDICTIONS AND COMMENTS
x	gradient	
GRADIENT RULE		Grad =

EQUATION $y =$		PREDICTIONS AND COMMENTS
x	gradient	
GRADIENT RULE		Grad =

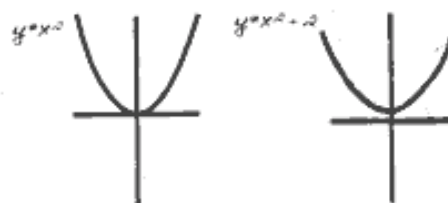
Any other theories, comments or questions:

Figure 2

During the following day's single lesson we were unable to use the computer room due to a timetable clash and so I asked the class to consider an algebraic justification of some of the results they had obtained. Recalling the general expression for the gradient of a line between pairs of points, I gave them the co-ordinates of two points on $y = x^2$, namely (x, x^2) and $((x + h), (x + h)^2)$. I noted that h represented the difference between the x -values of the points and that h had been one of the columns of their spreadsheet; I then waited to see what occurred. I hoped that they would remember from the spreadsheets they had written the day before that the values in this column had been halving each time and that they had decided upon their answer for the gradient at a particular point by looking at the limit of the sequence of gradients as the difference in x co-ordinates approached zero. Most were able to do so and realised that this meant that the value of h in the algebraic expression would need to approach zero for the gradient of the line to approach the gradient of the curve at the fixed point (x) . Also some pupils succeeded in going on and explaining algebraically the more complicated result for the gradient of $y = x^3$, (see Figure 4 for an explanation by Tim) which involves realising that any term multiplied by h can also essentially be ignored in the final answer as h is approaching zero.

Investigating the relationship between a gradient and the equation it is formed from.

We started studying the equation $y = x^2$ and found, by noting the gradient with different values of x , that the gradient = $2x$. We then found that, even if a number was added i.e. $y = x^2 + 2$, the gradient was still = $2x$. This is because, if the graphs were drawn, the gradients would be the same, but the line moved up i.e.



However, if a letter (x) is added to the $y = x^2 + 2$, the gradient changes.

Figure 3a Rose's work

By finding values for points on different curves, e.g. x^2 , x^3 and $2x^2$, it was possible to come up with a formula that would give the gradient of any point on a curve. This formula is:

$$nx^{n-1}$$

where n = the power to which x is,
 v = any number in the equation of the curve before x , e.g. $2x^2$
 x = value of x .

Example: $2x^2$ the gradient formula would be $3 \times 2 \times x^{3-1}$ or $6x^2$.

Figure 3b Peter's work

Gradients of Curves

During the investigation, a number of different formulas were chosen. For the first section, the no. of x^2 's was changed. Those used were x^2 , x^3 , x^4 and x^5 . For each one the nos. 1, 2, 3, 4, 5 and 6 were chosen as point on which the gradient would be found. These are the results:-

Graph	Gradient	after the sixth gradient was found in each graph, a prediction was made and they were all found to be correct.
x^2	$2x$	
x^3	$3x$	
x^4	$4x$	
x^5	$5x$	

From this we worked out a general formula by noting that the power of x for each graph was equal to the number in front of the x in the formula and was one more than the power of x in the formula.

Therefore:-

Graph	Gradient
x^n	nx^{n-1}

The second part of the investigation dealt with variable nos. of x 's after the x^n . For this we chose $x^2 + x$,

$x^2 + 2x$, $x^3 + x$, and $x^3 + 4$. These are the results following the same procedure as above:-

Graph	Gradient	
$x^2 + x$	$2x + 1$	The single figure added to the previous results is
$x^2 + 2x$	$2x + 2$	the amount of x 's of
$x^3 + x$	$3x^2 + 1$	the end of the formula
$x^3 + 4$	$3x^2 + 4$	of the graph.

From this an easy general formula was found and when tested was found to be true:-

Graph	Gradient
$x^n + zx$	$nx^{n-1} + z$

Further investigations could be carried out to find an even more general formula involving a variable number of x 's or having some x^m 's.

Figure 3c Oly's work

Gradients of Curves

Last week we discovered that the gradient of $y = x^2$ is $2x$

Gradient = $y_2 - y_1$

$$\begin{aligned} & x_2 - y_1 \\ & = (x+h)^2 - x^2 \\ & = (x+h)^2 - x^2 \\ & = x^2 + 2xh + hx + h^2 - x^2 \\ & \quad h \\ & = 2x + h^2 \\ & \quad h \\ & = 2x + h \end{aligned}$$

As the gradient is at a point h is so near to 0 that it can be ignored, similarly when $y = x^3$ the gradient is $3x^2$

$$\begin{aligned} \text{Gradient} &= (x+h)^3 - x^3 \\ &= (x+h)^3 - x^3 \\ &= (x^3 + 3x^2h + 3xh^2 + h^3) - x^3 \\ & \quad h \\ &= x^3 + 3x^2h + 3xh^2 + h^3 - x^3 \\ &= 3x^2h + 3xh^2 + h^3 \\ & \quad h \\ &= 3x^2 + 3xh + h^2 \end{aligned}$$

Again h is virtually 0 \therefore it and anything multiplied by it can be ignored.

Figure 4 Tim's work

As the first computer based session had gone so well and some groups had made so much progress, I was "forced" to set some much more "complicated" functions for consideration in the second session, namely $1/x$ and $x^{1/2}$. Because the gradient of these curves at integer values of x does not come out as an integer, this meant that the pupils now had the more complex task of identifying which decimal a particular sequence of decimals was tending to and then finding the associated fraction. This proved to be much more time consuming and indeed it was only when I suggested limiting the values of x being used to consider $x^{1/2}$ to square numbers that any significant progress was made. (See Figure 5 for the results of Tim.) An interesting point about the results for these functions is that no-one, even after obtaining the correct formulae, seemed to

Having worked out a general formula of nx^{n-1} this week we went on to see if things like $x^{0.5}$ and x^{-1} fitted the pattern basically they didn't

with $x^{0.5}$ or x
 we got the following results

x	Grad
1	$1/2$
9	$1/6$
16	$1/8$
25	$1/10$
36	$1/12$

which lead to the formula grad $\frac{1}{2\sqrt{x}}$
 ie $x = 16 \quad 16 \times 2 = 8$
 and the reciprocal of $1/8$

With x^{-1} the results were

x	Grad
1	$-1/1$
9	$-1/9$
16	$-1/16$

for which the formula is obviously grad $= -1/x^2$

Figure 5 Tim's results

make the link with the general formula they had obtained previously for x^n . In fact the more usual comment, as seen in Figure 5, was that the formulae they obtained did not match the previous pattern. This, I discovered when I questioned one pupil more closely, was mostly due to a confusion over the equivalent use of multiplying by a fraction and dividing by a whole number in the case of $x^{1/2}$ and a lack of thought in the case of $1/x$ as once one formula did not, so they thought, fit the pattern they did not look too closely at the other. The final problem I set was to take the basic ideas further by considering simple examples of the chain rule, namely $(2x+1)^n$. The results from Katherine (shown in Figure 6) are very interesting as they show clearly how she went about finding her formulae, by simplifying original results through factorisation and then noticing formulae for the factorised expressions.

The main difficulties I found in working with computers to study initial calculus were matters of logistics. The computer room in which we were working contains only 16 machines and so, as my group contains 30 pupils, there were usually two people to a machine. This in itself is not a disadvantage, encouraging as it does mathematical conversation and interchange of ideas between the pupils. However, I felt that because the cramped environment made it hard for me to circulate freely, some pupils were allowed to avoid engaging fully in the activity.

We used the spreadsheet to find the formula for the gradients of more curves. We tried x^2 and got

$2x$ which is another way of writing $2x^1$. We also tried x^3 and got $3x^2$ which is the same as $-x^{-2}$.

We also wanted to find out what the general formula for the gradient of a curve $(x+1)^n$. We tried $(2x+1)^2$ and got:

x value	gradient	which is	so got formula
1	12	4×3	
2	20	4×5	
3	28	4×7	$4(2x+1)$
4	36	4×9	
5	44	4×11	
6	52	4×13	

For $(2x+1)^3$ we got

x value	gradient	which is	which is also	so got formula
1	54	6×9	6×3^2	
2	150	6×25	6×5^2	
3	294	6×49	6×7^2	$6(2x+1)^2$
4	486	6×81	6×9^2	
5	726	6×121	6×11^2	
6	1014	6×169	6×13^2	

For $(2x+1)^n$ we worked out that the formula would be

$2 \times n(2x+1)$ for $(2x+1)^2$ it was $4(2x+1)$
 $(2x+1)^3$ it was $6(2x+1)^2$
 was $2(2x+1)$
 2×2
 for $(2x+1)^2$ it was $4(2x+1)$
 2×3

Figure 6 Results from Katherine

Having said this, the amount of learning that this project produced for most pupils in such a short period staggered me. After only two sessions, the entire group were able to predict with total confidence gradient functions for any polynomial function I cared to mention and many had progressed beyond that. Whether the results will be

remembered in the future and whether the pupils will use their results in a non-computer based situation is another question but the speed with which they established results was amazing. Indeed the pupils seemed much quicker in spotting formulae for individual cases and for the general algebraic results than I had noticed previously when carrying out investigations in class. The speed with which the computer was producing results for them seemed to make them want to think harder about the problem so they could obtain formulae quickly and move on. I think the very open nature of the problem allowed some pupils really to accelerate their own learning. The quickness with which my pupils were able to make generalisations matches what Tall (1985a) found in his study of methods of introducing differentiation although he used an entirely graphical approach with the computer. Another reason I think the group managed to grasp the concepts of calculating gradient functions so rapidly was that they were not being confused by any type of new notation and so were free to experiment. This echoes the sentiments of Orton (1985) who theorised that carrying out experiments of this type would lead to general principles being discovered without "elaborate symbolism" being introduced too early. However, David Tall has pointed out (Tall, 1985b) that when removing possibly confusing notation like dy/dx from the beginning of a calculus course until initial understanding of the mechanical processes of calculating gradients of curves is obtained, it is vital that understanding of the notation itself and how it is linked to the quotient it represents is established as soon as possible after this. This would be my next step, were I to continue teaching calculus to this group of pupils.

The speed at which the majority of the pupils grasped the ideas of gradient functions and the rules for their calculation in the case of polynomial functions makes me feel that this method for teaching initial differentiation has many advantages. I would like to think that when I carry out the same process with a much smaller A-level group and link it to the usual theoretical work and practice and applications of the concepts, the students will realise that the ideas being covered are vital to them at that moment and not, as was the case with my Year 10 group, at some unspecified time in the future, and the performance across the group will be more consistent.



David Crawford now works at Leicester Grammar School. This work was carried out in 1996/97 while he was teaching at Bristol Grammar School, as part of his Master's degree at Bristol University.

Bibliography

- English, R. (1993) *Introducing Spreadsheets in Mathematics. Mathematics in School*, Vol. 22(5), Pp 38-40.
- Orton, A. (1985) *When Should We Teach Calculus? Mathematics in School*, Vol. 14(2), pp 11-15.
- Tall, D. (1985a) *The Gradient of a Graph. Mathematics Teaching*, No. 111, pp 48-53.
- Tall, D. (1985b) *Tangents and the Leibniz Notation. Mathematics Teaching*, No. 113, pp 48-51.

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