

Mathematical thinking with dynamic geometry

Mark Boylan and Hilary Povey

Twenty-five school students from year 10 and two short afternoons. What mathematics should we do?

We had discussed a number of things the possibility that Mark's year 10 top set might come to the Mathematics Education Centre. At that time he taught at an inner city school with a predominantly white and almost wholly working class intake. He had been working with this group of students since the beginning of year 9 and had pushed them (and himself) to experience teaching and learning styles that positioned the students as the makers of mathematics. Mark was experimenting with the idea of 'hinge moments of learning', times when there seem to be qualitative shifts in our thinking. It seems that these hinge moments – when we swing around, find a new direction for our thinking, see things from a different angle – are likely to be provoked by being placed in novel situations and that coming to the University to do some mathematics might provide such moments for these students. Currently, new technology also seems to be a source of such opportunities. Thus the decision to bring Mark's students to the Centre and for Hilary to offer some work with software un-

familiar to them. The students' response during the two visits and the reports they wrote were very positive and we hope that something of what we did will 'work' for other teachers and students.

We began with a brief but explicit introduction to the nature of the dynamic geometry software that the students would be using. Hilary spoke about the objects that the software made available to the user – points, lines and circles – and invoked the simple example of two lines crossing at right angles to describe the difference between drawing and construction. If the lines have only been drawn, they may appear on the screen as right angles but, when you use the mouse to get hold of one of the lines and to drag it about, you can make that perpendicularity disappear. If on the other hand you have constructed the lines to be at right angles, then, no matter how you drag the screen objects about, the lines remain perpendicular. She tried to exploit the students' familiarity with computer drawing packages to help them grasp the distinction. We used the idea and vocabulary of mathematical construction and constraint as we interacted with the students working on the computers and some were later

also able thus to conceptualise and articulate their work. They spoke, for example, of having to 'connect the shapes mathematically'. Introducing such concepts explicitly with appropriate language was worthwhile: given more time, we would spend longer on this introduction and illustrate the words spoken with dynamic computer images.

We then set the first task. We showed them the kaleidoscope image on the OHP and invited them to explore making such a construction of their own. We were using *Geometry Inventor* and we drew their attention explicitly to the point menu, the polygon menu and the transformation menu.

In an important sense, the mathematics involved was not new to the students

The actual work we did at [the University] wasn't very different from the normal work we do ... We did use more sophisticated equipment but the basic work was almost the same

Matthew

but many reported *experiencing* it as different.

I learnt about maths of how you can use a computer to solve problems that you can't do in your head or using pen and paper

Joseph

[It] differed from what we do at school because the mathematics that we usually do at school contains mostly numbers

Zoe

... at the university we looked at shapes, patterns and lines whereas at school we tend to mostly do sums and equations, eg trigonometry, algebra, etc.

Patrick

However, their familiarity with the formal mathematical content addressed by the activity provided an environment within which they could confidently come to terms with the software. We asked them to print out some image(s) which pleased them and to take them back to school where they would write a little about what they had done and what they had learned. This material was brought back on the next visit and incorporated during the afternoon into a class display at the head of the stairs. Response to this was predictably gratifying but a number reported on

the liberation of not having to write everything down but instead being required to discuss, to invoke memory, to create and manipulate images.

I learnt about the University that they discuss things more than writing things down

Joseph

In school we do all our work on paper

Roger

... the work that we did involved more thinking and remembering what we had done, at school we usually write down everything that we learn or have learnt in the past

Zoe

The work we did there was quite challenging and I enjoyed it a lot. I enjoyed puzzling things out and trying my ideas. I also enjoyed being part of the 'group' and knowing that I was there to not just work on my own but to work with someone who I could talk to, work with and relate to. It also felt good to be able to talk to other people about my work ...

Joanne

I also learnt that maths isn't just writing, there are lots of practical things you can do ...

Patrick

... but the best thing was that we didn't have to do a lot of writing!

Matthew

These responses were thought provoking for us. Studies have shown¹ that students from working class schools spend a significantly greater amount of time than other students writing. The approaches to learning mathematics that the students had described and valued had been an important part of the way Mark had tried to work with this class. Nevertheless, the unspoken realities and culture of school life nudged him in the direction of 'write it down'. When reflecting on this Mark identified a strong fear that, if there is not a written record of work done, then the work will be less valid. He also recognised how this displays a lack of confidence that students will really learn more through discussion: they had better have a written record to help them 'revise' in case the content is not learnt. Is it the case that when using IT we feel more comfortable with the lack of 'writing down'? Perhaps we

feel that solutions to problems that our students type, construct or that simply appear on the screen do not need a permanent record?

We wonder if the current emphasis in the National Curriculum on teachers' record keeping, evidence, inspection and testing is a pressure away from the oral and group work these students so enjoyed. When we want students to engage with mathematical thinking and its reporting, perhaps we should more often say 'be prepared to talk about it'. It is ironic that the students who were critical of their usual diet of 'writing things down' were much more enthusiastic when writing their own record of the visits: here the process of writing was a creative individual act.

As the students noted, the mathematics on the second visit was much more demanding. We showed, again on the OHP, the figure of two intersecting circles and triangle³ and asked them to *construct* the figure so that it did not come apart when they dragged it about. As expected, this was challenging for all the students. We had a number of teacher education students in the classroom who had not worked on this problem before and were not themselves particularly familiar with the software but without their support the session might have foundered. However, none of the pairs of students was completely lost and several worked through to a solution. The activity appeared to be engaging and to provoke mathematical discussion, argument and reflection. Hilary visited the school after this session and spoke with a group of four students about their experiences.

Lucy The first time we did it, it didn't stay did it? You could drag it and it would make all different triangles, it would make isosceles and everything but then, what did we do? We thought it had to be at the middle, on the middle of the other so when we'd done that it were all right.

Peter We did the same as them and it all fell to bits [laughs]

Lucy [laughs] That's what we did first time

Peter And then we couldn't work out how to do it so we asked them and they showed us how to do it

Lucy We showed them that the point had to be the middle with the same radius and then the other point had to be the other radius

Hilary Why do you get an *equilateral* triangle when you do that?

Lucy Because it's the radius and they've all got the same, it's the radius

Those who found a solution were asked if they could *prove* that the triangle must be equilateral. We asked, 'How can you successfully argue that it must be what you say it is?'. We pointed out that the position, orientation and size of the construction could be varied by dragging but that chosen mathematical construction of 'equilaterality' remained unchanged. Then we asked, 'Could they build an isosceles triangle? Or a right angle triangle?'

A variety of ideas were tried but time constraints meant that many were interrupted before they came to fruition. Lucy and Tina had decided to approach the isosceles triangle problem by drawing a pair of circles not constrained to be the same size. They recollected.

Tina We had to make the circle wider, like there had to be a bigger space like ...

Lucy ... one circle were bigger weren't it, one circle were bigger than the other one

Hilary Do you know why it turned out to be an isosceles triangle? Why wasn't it just any old triangle?

Lucy ... because these two are the same distance because they are both going from the middle so they are the same distance apart.

Peter It's the radius

Peter and Darren described having tried a different approach. Talking about it afterwards, they were able to draw on an idea that had come up later in the session, even though the vocabulary escaped them, the idea of the perpendicular bisector of a line.

Darren We tried using just one circle instead of two ... what we was trying to do was seeing if we could make an isosceles triangle by, we drew a circle then we put two points down near the bottom equally apart from each other and then put another point at

the top above the points and then connected the lines together but it fell apart. Perhaps we could have tried it like them.

Hilary Let's stick with your idea for a moment. If you draw these two points and join them up, where has this point got to be?

Darren Directly in the middle of the other two

Hilary So how could you do it?

Lucy Oh that other thing the, the those lines, oh oh (sketching in the air)

Peter That line point connected oh oh (nodding)

Hilary I'll tell you the name 'cos it's the first time you'd ever heard it. It's called a perpendicular bisector

Peter Yeah!

Lucy That's it!

Hilary Why would that work?

Lucy 'Cos you could fix it directly at the point (points to line crossing circle) and then spread it up to wherever you wanted it

Peter Yeah!

Darren We could have done with knowing about that perpendicular bisector earlier!

Hilary had introduced the whole group to the construction of a perpendicular bisector at the end of the triangle activity, suggesting that they might want to make use of this construction in attempting the next task. With hindsight and with more time available, it would have been useful for the students to return to the triangle task after some group discussion both about possible strategies and about relevant constructions.

We started the second half of the afternoon away from the computers. The students sat in groups around tables on which there was a supply of tissue paper circles, geostrips, plain paper, pencils and pairs of compasses. The students were asked to use the materials in any way they chose to make a square. Various ideas emerged and Hilary

asked several of the students to share their results with the class. As they did so, she asked the class to consider what property of the square the particular solution was exploiting. Suggestions included using geostrips, which drew on the properties of four sides the same length and one right angle, and a variety of ways of folding the circles. For example, Darren used two paper circles folded identically and then brought together.

Darren What I did with the paper tissue, I made a square out of it, I folded it in half and then I folded it into quarters, then put the two rounded edges together and it made a square.

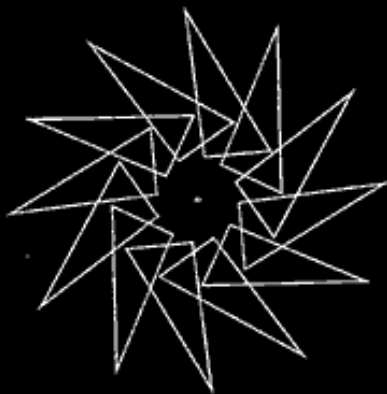
When Darren explained why this had produced a square, he drew attention first to the pair of opposite right angles which had been generated, then to the way in which these had been brought together to produce two more right angles and finally to the fact that, because the circles were the same size, the resulting sides were each half the diameter and therefore the same as each other. Tina had used a different series of folds, exploiting different properties of a square.

Tina I folded it in four and then again and just folded it straight on the diagonals ... I folded it and then folded it again and then the edges of the diagonals gave me the corners to fold on

The students were then invited to return to the computer room and choose any polygon they liked to construct. Unfortunately, there was not enough time for this work to develop satisfactorily but all the pairs set to work with a will and had no difficulty in setting themselves a task and beginning to tackle it. Naturally, this is substantially as a result of patterns of working which Mark had set up working with the group over time. Nevertheless, we also felt that the dynamic geometry environment had been motivating and had encouraged them to think and to discuss mathematically.

Afterwards Mark asked the students what they had liked about the visits, what, if anything, they had found different and what they felt they had learned. All but one of the students were very positive and their comments ranged over a wide variety of subjects. They had definitely appreciated the opportunity to work with computers.

Kaleidoscopes



This kaleidoscope was made by rotating a triangle around a point.

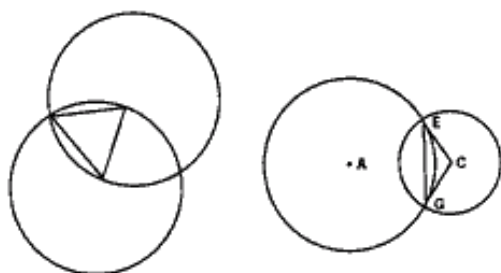
Design an interesting kaleidoscope of your own.

Make the kaleidoscope move by dragging your original shape.

You can drag one of the corners or the complete shape. You can also drag the central point.

You might get some more interesting results if you use reflection and enlargement as well as rotation.

From Dynamic Geometry, NCET.



I liked it because we don't do owl like this at school ... We don't like usually get any software, we have to do it from books and drawing things

Tina

Some specifically commented on how this had made the mathematical experience different.

I liked doing them circles best, the ones with the triangle, we thought about trying to do a scalene triangle but we didn't have the time. We thought that were good when we were trying to work out about why it did that (the equilateral triangle) ... it took us more than once to try and work out the first time and then once we'd got that we could like go onto other things ... I liked it because we had to experiment ... if I have to do it on paper I'll try and avoid experimenting

Lucy

It took a little getting used to the program because the thing is we've never used anything like that before ... I've used drawing packages before but with this you had to find the mathematical connection between the shapes and it made me think a lot more about how shapes had an effect on maths

Peter

Finally, there seemed some evidence that both the University experience and the dynamic geometry environment had provided some of the 'hinge moments of learning' that Mark had set out to provoke.

It makes you look at [maths] a different way

Peter

I also learnt a lot about myself. I learnt that I can work with a partner and in groups to solve problems, and I can work on a puzzle until it is solved, correcting any mistakes I make and learning from them.

Matthew

And Mark's response to his students' writing about their work shows that perhaps it was a hinge moment for him as well.

Visits to the Maths Education Centre

Thank you for your piece of writing about your work at the university. I asked you to write about your expectations before we went, about your

impressions and about what you thought you had learnt. These are my thoughts on the visits.

I organised the visits for three reasons. Firstly, I wanted you to experience the atmosphere of a university so that you could make an informed choice in the future about whether you might like to go to one. Secondly, I wanted to give you the chance to explore mathematics in a different environment and in a different way. Because of the pressure of time, class sizes and lack of resources, often as a class I have not been able to work with you in the way I would most like. Ideally I think you should be able to explore and discuss your mathematics in a more open way. Thirdly, I think the most memorable times at school are when you go on visits or are doing something unusual. You can learn more, particularly about yourself, when you do something new or for the first time.

I was very proud of the way you worked on the two visits and of the display that you produced. The workers at the centre made very positive comments about you.

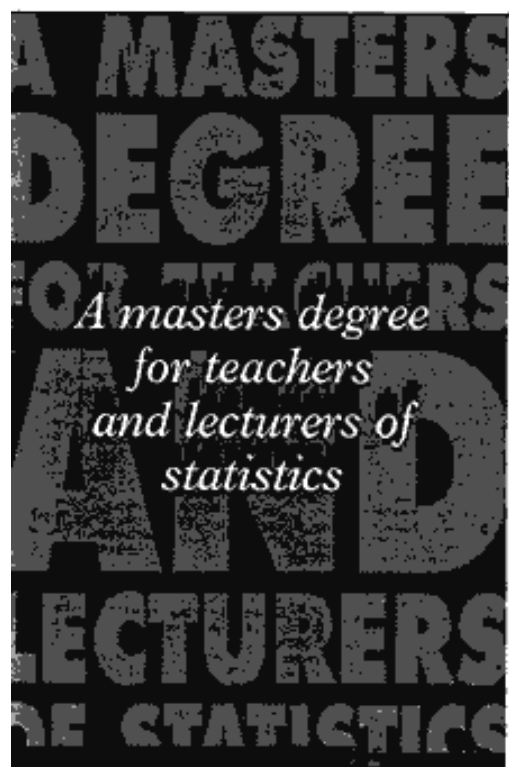
I learnt a lot during the two visits and from the written work that you produced afterwards. It has changed the way I teach in number of ways. You found the reality of the university atmosphere very different from what you expected: this reminded me of the cultural differences between students and teachers in schools.

I found that the geometry package could be a very powerful tool for learning. And I was reminded of how important it is to let you try and solve problems for yourselves rather than simply telling you how to do something.

Hilary Povey works at Sheffield Hallam University. Mark Boylan worked at Chaucer School, Sheffield when this article was written. He now works at Northcliffe School, Doncaster.

References

1. *Classroom approaches to teaching and learning: the social class dimension*, Sally Brown, Sheila Riddell and Jill Duffield, reported in TES, 4 October 1996, p7.
2. Based on an idea from a conversation with Julie-Anne Edwards, Rillgers Park School, Southampton.



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