NOTATION

Here Anne Watson explores the importance on notation in supporting children's understanding of mathematics.

This year the topic for the Annual Institute of Mathematics Pedagogy 2008* was 'notation'. We worked for three days on a range of mathematical tasks in order to gain a deeper understanding of the role of notation in mathematics education. Our general conclusions were that while mathematical notation can be powerful it can also be obstructive. This is partly to do with individual perception, understanding, and familiarity but at the Institute we were able to identify other issues related to notation:

A notation tends to be powerful when:

- it is grounded in well-established conventions
- it simplifies, identifies key features and ignores irrelevant/redundant ones
- it reveals more structure than was used to devise it
- it offers the learner more possibilities than the original problem
- the meaning of the notation and the purposes of the learner are related
- it facilitates leaving concrete contexts aside and working in the abstract.
- it allows the learner to 'let go' of what it is representing
- it is generative in that the learner can then see the situation and its interrelations differently
- the learner knows some examples and instances of what it represents
- it is extendable and/or consistent
- it means the same thing to all learners
- it enables communication with self or other learner
- it becomes a means of expressing and crystallising thoughts

A notation tends to be obstructive when:

- it is not closely related to, or it even obscures, structures or processes
- it is ambiguous
- it is at odds with the situation as the learner sees it
- the learner loses a grip on the situation it represents
- it limits what you can do in a situation
- the learner is not comfortable with the context in which it is being applied
- it acts as a veil between you and what you are thinking about
- it imposes new rules on the learner that are not in the original situation
- it tells the learner what the range looks like, but not what the transformation from domain to range actually does.

Examples of notation that may obfuscate meaning or confuse learners:

- P(A/B)
- 11 : 4
- On a calculator, is -3^2 $(-3)^2$ or $-(3^2)$
- Notation for subtraction and negative numbers e.g. $- -3$ and $- {}^{-}3$
- Bar notation: $\bar{2}.356$ means $-2+0.356 = -1.644$ so what is $\bar{2}.356 - \bar{3}.217$?
- Vectors: when free and when fixed?
- Use of upper case and lower case letters as points, forces, vectors, lengths etc.
- $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ $\begin{Bmatrix}1 & 0\\0 & 1\end{Bmatrix}$ $\begin{vmatrix}1 & 0\\0 & 1\end{vmatrix}$: when is it important what sort of brackets you use?
- Most aspects of logarithmic notation were reported as sources of confusion
- Misconstruing $\sin^2(x)$ as $\sin(\sin(x))$, which might otherwise be $\sin^{(2)}(x)$; misconstruing $\sin^{-1}(x)$ as $1/\sin(x)$ or cosec (x) rather than the inverse of the sine function
- o, a, h used for sides of right-angled triangles are labels, not unknown or given lengths.
- in algebraic expressions, inconsistency between use of $=$ and $\equiv$
- 1,000; 1 000; 1.000 depending on your cultural origins
- Is f' a transformation related in some way to f, or a differential?

For some mathematical ideas there are choices of notation that afford different approaches. For example, for derivatives one can choose from dy/dx; $f'(x)$; $\overset{g}{x}$; but the final kind is hard to use with higher derivatives, and the first kind invites a rather casual attitude to 'cancelling'. For vectors one can choose $\boldsymbol{\nu}$, $\overline{v}$; $\boldsymbol{\gamma}$ etc.. For rational numbers one can use a/b or decimal notation whereas for fractions and irrationals the decimal notation might not give an accurate representation. These illustrate the

* IMP is run by John Mason, Malcolm Swan and Anne Watson. If you would like to go on the mailing list for information please contact j.h.mason@open.ac.uk

need to understand the limitations of notations in order to make sensible choices. (We could add to this Derek Ball's observation in ***MT211***, p.12 that there is deliberate ambiguity in the way we use the sign for 'square root' and what we mean by 'square root'.)

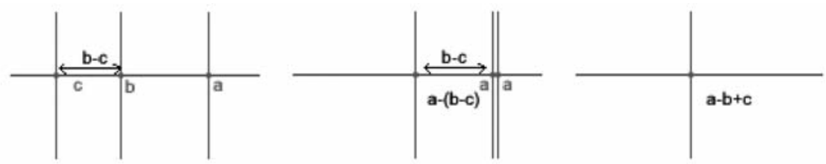
To get to these insights we engaged in a number of notation tasks. One of these was:

> Imagine a number line with another 'acetate copy' of it on top. Imagine rotating the copy through 180 degrees about the point 3. Where does 5 end up? 1? –1? Generalise; extend to compound rotations. Denote by R_a the rotation through 180 degrees about the old point a on the original number line.

The point of the exercise is to experience the power of the subscript notation which succinctly captures the underlying arithmetic. R_a can be expressed algebraically as acting on any number x and sending it to $2a - x$. This is an expression of various ways of 'seeing' the rotation. For example, you can translate a to the origin, then rotate about the origin, then translate back again: x is sent to $(x - a)$ which is sent to $-(x - a)$ which in turn is sent to $a - (x - a)$.

In general, R_bR_a turns out to be the same as $T_{2(b-a)}$, translation to the right by $2(b - a)$ sending x to $2b - (2a - x)) = x + 2(b - a)$.

$R_cR_bR_a$ can be seen either as R_c following $T_{2(b-a)}$ or as $T_{2(c-b)}$ following R_a. In either case the result sends x to $2c - (x + 2b - 2a) = 2(c - b + a) - x$, which is rotation about the point $c - b + a$. Some people find the diagrams below helpful for this.



$R_cR_bR_a$ same as same as

As one participant said, "I could visualise the effect of a single transformation or a pair, but I let go of the attempt to visualise for three transformations, $R_cR_bR_a$, where I was content to accept the algebraic proof (of my neighbours) that this is equivalent to R_{c-b+a}."

Implications for teaching

Using notation to represent an idea that has first been met in a practical context, or through an example, is a very different experience to using a notation to express abstract ideas that cannot be accessed in any other way. The first of these could be called 'denoting'. For example, using a positive integer to represent the amount of sweets I have is a different experience to representing a negative number on a numberline when I do not yet understand negative numbers. We wondered if, as teachers, we pay sufficient attention to helping students understand the difference between using notation to represent what we already know, and exemplifying a notation in order to understand a new abstract idea. In mathematics, as in other sophisticated symbol systems, notation makes more things possible than can be created without notation. Mathematical notation enables us to contemplate new objects as well as denoting old objects.

A related issue about notation is that those which are most useful for working with may not be the most useful for expressing generality. For example:

> To fold a strip of paper in thirds, first fold any length at the right-hand end and call it the first approximation to a third. The left-hand end now represents two-thirds; fold it in half to get the second approximation to a third. The right-hand end now represents two-thirds; fold it in half to get a third approximation to a third and so on ...[1] To explain this and to give each other instructions people typically devise a notation or diagram which records the folds. However, to express what is happening generally the following notation is often offered:
>
> $a_1 = (1-a_0)/2$
>
> $a_2 = (1/a_1)/2$
>
> $a_3 = (1-a_2)/2$ etc.

In the Institute we experienced all of these tensions and contradictions for ourselves and began to see that many of the difficulties learners have may be associated with not being given enough time, and sufficient different experiences, to make connections between notations and their meaning, particularly when the meaning cannot be modelled except through notations.

It also seems that having to relate two different notations for the same idea is an important component of conceptual understanding; having to toggle between diagrammatic and symbolic notations encourages the development of inner language to make the transformation, and Descartes knew a thing or two when he suggested combining algebraic and graphical representations.

Some of us when thinking about our future teaching resolved to:

- involve pupils in the invention of notations for consistent structures to help them to appreciate that notations were invented, and why they were invented.

1 This task comes from Trevor Fletcher and was introduced to us by, we think, Geoff Faux in an ATM conference session about 14 years ago.

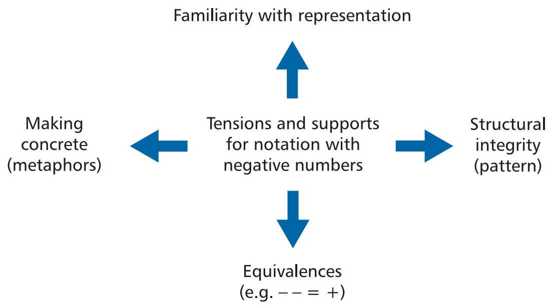
- be aware of mistakes /misunderstandings that arise frequently because of notations, and to construct activities that deal with these head on.
- not attempt to shield pupils from difficult changes of notation in different contexts. We need to support them explicitly in dealing with this, e.g. $2a/2b = a/b$ but 23/25 does not equal 3/5. Another example is that the '=' has different meanings in $3x - 6 = 12$ and $3x - 6 = 3(x-2)$
- appreciate that students are capable of dealing with different meanings and different notations because they do so in spoken and written language, e.g. write/right/rite and row/row.
- not to become so attached to our own notations and models for teaching as to sometimes avoid others.
- recognise that our pedagogy needs attention, not trying to simplify notations or trivialise meanings.
- "… no longer refer to brackets as 'crab claws' as this obscures meaning and doesn't help when there are more terms or more brackets".

A further challenge for our pedagogy arose when we thought about our students' future encounters with notations. Why not use notations that imply 'there is more' rather than limit students' experience to what they need at the time? For example, in some primary classrooms the displayed numberline starts at zero. Why not always use number lines that extend below zero so that negative numbers do not require an alteration of the model, but can be contemplated from the model we already have? We can use the existing model/notation when a need arises for negative numbers. Similarly, if we always use the grid method for multiplying two 'numbers' that can be expressed in chunks, we can use this existing model/notation when multiplying numbers which we can only express in chunks, such as irrationals or complex numbers. In this way, extending the uses of a good notation becomes a mathematical habit. Extending this idea, we wondered if some notations which arise later in harder contexts can be introduced earlier when the context is more easily understood. For example, if all number machines were given a function name, e.g. FRED(x), then it might be easier for students to accept sin(x) as a function because they are used to the notation.

The case of negative numbers

When we applied our insights to negative numbers we first went down the route of wanting notation to be unambiguous. For example, is it helpful to have different signs for negative numbers and subtraction? Some curriculum approaches suggest using $^{-}1$ to indicate the signed number 'negative one'. Then we need to help students see why different signs are redundant and so why we only need to use '–'. Another approach is that children may be helped to learn conventions regarding negative numbers if they know that the relationship between mathematics and the world is not a straightforward mapping, and that notations are invented to express a 'complete' conceptual system, and not all inventions work perfectly in every context.

This diagram expresses some tensions students need teachers' help to handle:



Like many of the topics we study at the Institute, when we go back to our normal work we find more and more to think about in practice.

Pedrio Palhares, who comes regularly to the Institute, wrote the following piece as a reminder that notation isn't everything, and that many learners have to struggle in situations where the teacher is unaware of the tensions we have just described:

> *In the construction phase of mathematics, notation is used in order to make some progress on solving a problem. Notation may be personal and idiosyncratic as it is to serve as a tool. If badly chosen, it may be an obstacle. If very well chosen, it may be a good support to get to the solution of the problem.*
>
> *If and when the problem is solved, then notation has to be reconsidered. Is it confusing or otherwise inadmissible? Then a new notation has to be devised and used. It may be that a previous notation has its place and suits better. The objective of this phase is to communicate with peers, so some idiosyncrasy may be admissible but not too much. Qualities to be considered are: coherence, consistency, eliminating redundancy.*

Notation can change in the teaching phase, as now a new purpose (targeting a wider audience) exists. New qualities may now emerge, like: explaining, revealing, enabling extension to other situations, among others.

According to Brousseau (1997), teachers should recontextualize and repersonalize knowledge. So notation is not (or shouldn't be) all that is to be considered. A problem has to be devised (either the original one or one more appealing or more down to earth), a context, a representation – or possibly several. The context may now involve manipulation of objects, and pedagogical strategies have to be devised. It is possible, and many times it happens in practice, that teachers, pressured to acknowledge some learning has occurred, accept and promote an epistemological shift, from the content to be learned to the notation that supports it. Therefore students will learn how to use the notation instead of learning the concepts involved. Considering this, we may think that there should be a limit to how far the notation should be independent of the concept.

As for the learner, (s)he may find the notation an obstacle. That may happen because it doesn't show a clear connection to the concept, or rules are not clear, or confusion arises. Learners (depending on their level) should change notation if it comes to that point. But not all learners have such autonomy on their learning. They may get stuck because of bad choice of notation, or have a personal resistance to one notation. At this phase, apart from clarity, the desirable quality for a notation is that it becomes empowering for the student.

Contributors/participants: Aimee Bray, Papa Kofi Boateng, Fiona Cockerham, Sue Elliott, Roy Grice, Dietmar Küchemann, Neil Lennox, Chris Martin, John Mason, Pedrio Palhares, Jenny Piggott, Mark Richards, Leo Rogers, Lucy Sayce, Damon Vosper Singleton, Erika Stadler, Malcolm Swan, Jo Tomalin, Anne Watson.

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Brousseau, G. (1997). *Theory of Didactical Situations in Mathematics*. Dordrecht: Kluwer.

WORKSHOP JOTTINGS

It all happens at 'Sheffield Hallum' on a Saturday morning. Corinne Angier asks, "how about joining them?"

The ATM all attainment teaching and learning group has been meeting at Sheffield Hallam for several years now. It includes new and experienced teachers and lecturers from the university. Some have extensive experience of working with all attainment groupings and some have none. All of us are interested in mathematical activities which have the potential to engage learners and encourage them to develop their mathematical understanding. The format of each Saturday session is roughly the same. We work together on two or three tasks and discuss the mathematics. Inevitably we are sidetracked into discussions about our classrooms but by deliberately making the mathematics our main focus we achieve a reasonable balance. Over lunch our conversation tends to broaden out as we catch up with each other's news and debate wider issues in education. After lunch we usually manage one more task before people have to drift away. The following jottings are a personal recollection of our most recent meeting. The basic structure of working on some mathematics together could be reproduced in almost any slot from 10 minutes at the start of a team meeting to a whole day of CPD. It is my own view that teachers are very aware of their responsibility to know their students, reasonably well aware of the importance of pedagogical understanding but perhaps less conscious of an equally important imperative to know the mathematics in their curriculum resources. All are necessary preparation for teaching any class well and especially for meeting the challenge of all-attainment classes.

We were a larger group this time and I was reminded how scary it is 'doing' maths with strangers. No wonder so many children in the classroom feel uncomfortable. John introduced the first task which was based on a sequence of growing, spiralling squares whose edge lengths are Fibonacci numbers. He described how after several lessons where students investigated perimeters and areas one of his students decided to put the diagram