VISUALISING GEOMETRY

As a result of attending the weekend on moving images, Barbara and Derek Ball consider the balance between visualising and verbalising.

What a way to celebrate your seventieth birthday! The genius of Geoff Faux was to organise a course at Hawes End in the Lake District on visualising, to which 40 people came. Various people were invited to lead sessions.

Aidan Harrington's session reminded us that different people learn in different ways; some of us seem to prefer to learn through our eyes, others through our ears. This made us think about the difference in learning style between the two of us. On one occasion, Barbara suggested to me that there was no point in attending lectures when the lecturer distributed a perfectly good handout, to which I replied that it was a waste of time producing a handout when the lecturer gave a perfectly good lecture.

In Alf Coles' session we viewed a geometry film about a cardioid (if you want to give a posh name to the curve produced)[1]. Derek struggled to understand what was going on. After we had viewed the film, Alf invited us to 'reconstruct it' as a group by describing what happens. After a while, Derek started understanding what the film was about, and just at this point Anne Watson suggested that this kind of discussion might not be all that useful – we had all seen the film and so what was to be gained by describing it (or words to that effect)!

During Geoff's weekend we had three different approaches to visualising: In Laurinda Brown's session we were invited to visualise from given verbal descriptions. In John Mason's session we had a mixture of discussion (verbal description) and visual images to support our visualising. In Alf Coles' and Dave Wilson's sessions, we were offered just films, although afterwards we had the opportunity to reconstruct the films by verbalising what we saw.

In thinking about the balance between visualising and verbalising (and verbal reasoning), we began to appreciate more than before that geometric visualisation is greatly aided by what we know – if we know that the angle in a semicircle is a right angle then that is what we see – we do not need to pay much attention to it, although we might be invited to pay attention to the consequences of the right angle – whereas if what is being manifest is a property of a cardioid that is unfamiliar, then a great deal of attention is required. Reasoning can also be used, based on previous experience. We were visualising two arms of the same length rotating about a common centre. At one point, we were invited to keep one of the arms fixed and rotate the other and then consider the locus of the mid-point of the tips of the arms. For some people this triggered known geometrical results (different results for different people) which made visualisation pretty much unnecessary; other people seemed to see the locus pretty much as a straight visualisation of what was happening (perhaps with a bit of guesswork); others struggled with the image and wanted to make sketches. There is no such thing as pure visualisation starting with an empty mind, because none of our minds is empty.

At this year's Easter conference, Geoff Faux invited Barbara (and others) to work on what happens if you took the mid-points of the faces of a regular tetrahedron as the vertices of another tetrahedron – what is the ratio of the volumes? Barbara had drawn various sketches and found that they were not helping her to visualise the problem and had more or less given up. Barbara and Derek were discussing the problem one morning and Derek suggested that Barbara simply try to visualise it in her head. Much to her surprise Barbara found solving the problem in this way much easier than when she was trying to prompt herself with 'helpful' drawings. We even both went on to agree that the problem was essentially no different if the tetrahedron was not regular. Here again we were using visualisation, but also a good sprinkling of known triangle geometry, etc.

Dave Hewitt's session was not about geometrical images, but about powerful images which might be useful in thinking about various topics in the mathematics curriculum (see p6 in this issue). Dave offered us two examples: the 'Gattegno chart' for place value and Dick Tahta's suggestion of a

1 Go to www.atm.org.uk/mt205 to view Trevor Fletcher's film 'The Cardiod'.

point moving round a unit circle on a coordinate grid for trigonometry. We worked on thinking about other images that might be empowering in similar ways. One image that strikes us as powerful for similarity, enlargement, properties of triangles, and other shapes, is a triangle divided into four triangles by joining its mid-points. The idea of powerful images that it is every learner's right to be offered is one which we strongly endorse: it often seems important to us to be able to hold ideas firmly in our heads before we feel we understand them, and strong images can help us to do this.

Sometimes the powerful 'image' might be less a picture and more a unifying idea or theme. Using enlargement as a unifying theme has helped us to hold in our minds ideas concerning the geometry of the triangle – Euler line and nine-point circle – in ways that we have been unable to do so easily without it. Readers might be interested in this packaging up of these ideas.

Barbara Ball and Derek Ball run mathematics masterclasses.

Visualising the Euler line and nine-point circle using enlargement

D, E and F are the mid-points of the sides of triangle ABC. The triangle ABC is an enlargement of triangle DEF with scale factor –2. This means that the medians AD, BE and CF of triangle ABC are concurrent at the median point M (the centre of enlargement), which divides each median in the ratio1:2.

The circumcentre O of triangle ABC is also the orthocentre of triangle DEF. Using the enlargement it follows that the orthocentre H of triangle ABC lies on the line OM and that HM:MO = 2:1. HMO is the Euler line.

Call the circle through D, E and F the medial (or nine-point) circle of triangle ABC. By enlargement, the centre N of this circle lies on the Euler line and NM:MO = 1:2. This means that N is the mid-point of HO.

Since the radii of the medial circle and circumcircle of ABC are in the ratio 1:2 and since HN:HO = 1:2, H is a centre of enlargement for the two circles. Given that the circumcircle passes through points A, B and C, it follows that the medial circle passes through the midpoints of AH, BH and CH.

If H_A, H_B and H_C are the reflections of H in BC, CA and AB respectively, it can be established that H_A, H_B and H_C lie on the circumcircle. Hence the feet of the altitudes lie on the medial circle. This completes the proof that the medial circle passes through the nine points which are D, E, F, the mid-points of AH, BH and CH and the feet of the altitudes.