

From Archimedes to limits, Bob Burn

Part 1. The Spiral and its applications in the 17th century No limits here.

1. Is there a smallest positive number? or a greatest *negative* number? Argue by contradiction.

2. If $-\varepsilon < A < \varepsilon$, for all positive ε , what may be said about A ?
This theorem will be used repeatedly, and will be called “The Vice”.

What makes analysis so different from algebra or geometry is its repeated use of inequalities to obtain equalities.

3. [*Archimedean Order*] In two of Archimedes letters, he articulates a principle (also to be found in Euclid V. Def.4) that if two quantities are given, some multiple of the first will exceed the second. This axiom excludes infinitesimals. It allows the comparison of two line segments or two areas or two volumes, but not a ratio of a line segment to an area.

Use this axiom, with 2 (above) to show that if $-1/n < A < 1/n$, for all positive integers n , then $A = 0$.

Archimedes had neither algebra, nor negative numbers, so he did not express things this way. Without negative numbers, the two inequalities are established by independent arguments. If you find Archimedean Order awkward to use, you may be surprised to realise that you depend on it every time you want to say $(1/n) \rightarrow 0$.

4. Can you extend the result of 3 to show that when B and C are constant positive quantities, and $-B/n < A < C/n$ for all positive integers n , then $A = 0$?

5. If U is an unknown quantity, and K is a known quantity, and $-B/n < U - K < C/n$, for all positive integers n , prove that $U = K$.

This is the structure behind many of Archimedes’ determinations of areas and volumes.

6. [*The Spiral*] An example (a rather significant example) of how Archimedes used the idea of 5 is his quadrature of the spiral. He showed that the area within one circuit of the spiral $r = a\theta$ was equal to one third of the area of the circumscribing circle. We examine the part of the spiral from $\theta = 0$ to $\theta = 2\pi$. The radius of the circumscribing circle is $2\pi a$ and the circle is divided into n equiangular sectors. Within each sector, say from $\theta = 2\pi(i - 1)/n$ to $\theta = 2\pi i/n$, we compare that part of the spiral with the largest circular sector *inside* the spiral and the smallest circular sector outside the spiral to get:

$$\frac{1}{2}[2\pi a(i - 1)/n]^2[2\pi/n] < \text{portion of spiral} < \frac{1}{2}[2\pi a i/n]^2[2\pi/n].$$

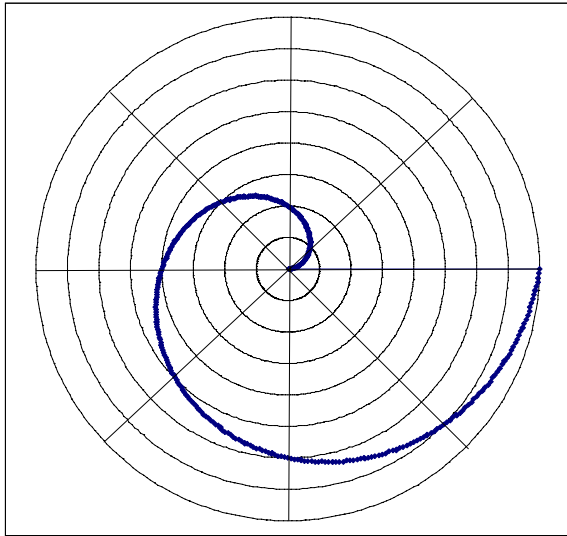


Figure 1. Spiral, circumscribing circle and sectors of angle $2\pi/n$, for $n = 8$. (The area of a circular sector is $\frac{1}{2}r^2\theta$ where r is the radius of the sector and θ the angle at the centre.) Adding the inscribed sectors for $i = 1, \dots, n$ we get,

$$\sum_{i=1}^n \frac{1}{2} \left(\frac{2\pi a(i-1)}{n} \right)^2 \left(\frac{2\pi}{n} \right) = \left(\frac{4\pi^3 a^2}{n^3} \right) (1^2 + 2^2 + \dots + (n-1)^2) < \text{area of spiral.}$$

Adding the circumscribed sectors for $i = 1, \dots, n$ we get,

$$\sum_{i=1}^n \frac{1}{2} \left(\frac{2\pi a i}{n} \right)^2 \left(\frac{2\pi}{n} \right) = \left(\frac{4\pi^3 a^2}{n^3} \right) (1^2 + 2^2 + \dots + n^2) > \text{area of spiral.}$$

Now Archimedes worked out the sum $1^2 + 2^2 + \dots + n^2 = (n/6)(n+1)(2n+1)$ in the middle of his proof. Using this result, the sum of the areas of the inscribed sectors is

$$4\pi^3 a^2 (1/6)(1-1/n)(2-1/n),$$

and the sum of the areas of the circumscribed sectors is

$$4\pi^3 a^2 (1/6)(1+1/n)(2+1/n).$$

So if S is the area of the spiral

$$4\pi^3 a^2 (1/6)(1-1/n)(2-1/n) < S < 4\pi^3 a^2 (1/6)(1+1/n)(2+1/n).$$

Now the area of the circumscribed circle is $\pi(2\pi a)^2 = C$, say. So, we set up a vice by getting

$$4\pi^3 a^2 (1/6)(1-1/n)(2-1/n) - \frac{1}{3}C < S - \frac{1}{3}C < 4\pi^3 a^2 (1/6)(1+1/n)(2+1/n) - \frac{1}{3}C.$$

However,

$$4\pi^3 a^2 (1/6)(1-1/n)(2-1/n) - \frac{1}{3}\pi(2\pi a)^2 = C(-3/2n + 1/2n^2)$$

and

$$4\pi^3 a^2 (1/6)(1+1/n)(2+1/n) - \frac{1}{3}\pi(2\pi a)^2 = C(3/2n + 1/2n^2).$$

So

$$C(-3/2n + 1/2n^2) < S - \frac{1}{3}C < C(3/2n + 1/2n^2),$$

and since $1/n^2 \leq 1/n$, we can say that $-2C/n < S - \frac{1}{3}C < 2C/n$.

Now this holds for all positive integer values of n , so using the Archimedean postulate,

$$-\varepsilon < S - \frac{1}{3}C < \varepsilon$$

for all positive ε , and it follows that $S = \frac{1}{3}C$.

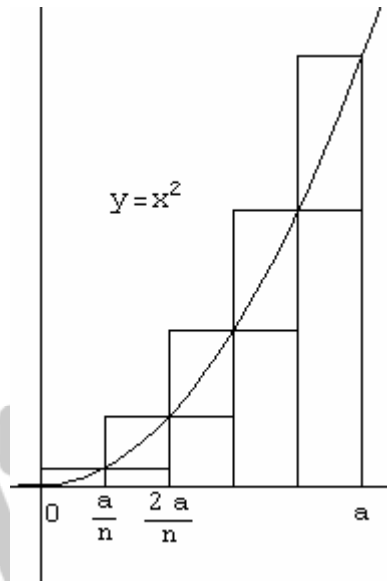


Figure 2. Parabola $y = x^2$, with inscribed and circumscribed rectangular strips

7. Use Archimedes' argument in 6 to obtain Fermat's quadrature of the parabola in 1636: the area bounded by x -axis, $x = a$ and the parabola $y = x^2$ equals $\frac{1}{3}a^3$. This can be found by working with rectangular strips, parallel to the y -axis, of width a/n , starting by calculating the area of the inscribed rectangles (like the inscribed sectors of the spiral), and then the area of the circumscribed rectangles (like the circumscribed sectors of the spiral). See figure 2.

Fermat had been extending Archimedes work on $r = a\theta$ to other spirals before he found that the method could be used for other curves.

8. Use Archimedes' argument in 6 to obtain Eudoxus' result (given by a much harder method in Euclid XII) that the volume of square-based pyramid = $\frac{1}{3}$ base area \times height. Work with square prisms parallel to the base of the pyramid and of thickness (height)/ n .

9. Use Archimedes' argument in 6 to obtain Eudoxus' result (given by a much harder method in Euclid XII) that the volume of a right circular cone is = $\frac{1}{3}$ base area \times height. Work with cylindrical discs parallel to the base and of thickness (height)/ n .

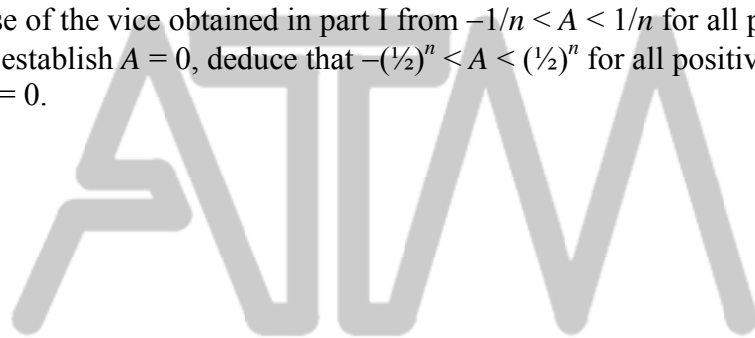
10. Use Archimedes' argument in 6 to obtain Fermat's quadrature of the 'higher parabola' in 1636: the area bounded by x -axis, $x = a$ and the curve $y = x^3$ equals $\frac{1}{4}a^4$. This can be found by working with rectangular strips parallel to the y -axis of width a/n . This needs $\sum_{i=1}^n i^3 = \left[\frac{1}{2}n(n+1)\right]^2$, which was known to the Arabs.

11. Having found the volume of a cone in 9, one can find that the volume of a paraboloid formed by revolving $y = x^2$ about the y -axis and cut off by the plane $y = R^2$ equals one and a half the volume of the cone with the same base and vertex at O . This was obtained by Archimedes, in his book *On Conoids and Spheroids*, by working with circular discs parallel to the xz -plane and of thickness R^2/n . The method is algebraically similar to the proofs above, though, rather than the sum of the squares, this proof needs $\sum_{r=1}^n r = \frac{1}{2}n(n+1)$.

12. Another result obtained by Fermat in 1636: the area bounded by the spiral $r^2 = a\theta$ and the line $\theta = 0$ is one half the area of the circumscribing circle. This also needs the Σr formula as in 11, above.

Part II. The Quadrature of the Parabola and Geometric Progressions.

1. The Principle of Archimedean Order appears in Euclid Book V, Definition 4
2. Use mathematical induction to show that $2^{n-1} \geq n$ for all positive integers n . Euclid of course did not state an inductive argument, but he certainly understood the idea in this case. Deduce that $(\frac{1}{2})^{n-1} \leq 1/n$, for all positive integers n .
3. Making use of the vice obtained in part I from $-1/n < A < 1/n$ for all positive integers n to establish $A = 0$, deduce that $-(\frac{1}{2})^n < A < (\frac{1}{2})^n$ for all positive integers n , also gives $A = 0$.



Association of
Teachers of
Mathematics

01332 346599

© ATM and others

copyright@atm.org.uk

The attached document has been downloaded or otherwise acquired from the website of the Association of Teachers of Mathematics (ATM) at www.atm.org.uk

Legitimate uses of this document include printing of one copy for personal use, reasonable duplication for academic and educational purposes. It may not be used for any other purpose in any way that may be deleterious to the work, aims, principles or ends of ATM.

Neither the original electronic or digital version nor this paper version, no matter by whom or in what form it is reproduced, may be re-published, transmitted electronically or digitally, projected or otherwise used outside the above standard copyright permissions. The electronic or digital version may not be uploaded to a website or other server. In addition to the evident watermark the files are digitally watermarked such that they can be found on the Internet wherever they may be posted.

Any copies of this document MUST be accompanied by a copy of this page in its entirety.

If you want to reproduce this document beyond the restricted permissions here, then application MUST be made for EXPRESS permission to copyright@atm.org.uk

*This is the usual
copyright stuff -
but it's as well to
check it out...*



The work that went into the research, production and preparation of this document has to be supported somehow.

ATM receives its financing from only two principle sources: membership subscriptions and sales of books, software and other resources.

Membership of the ATM will help you through

*Now, this bit is
important - you
must read this*

- Six issues per year of a professional journal, which focus on the learning and teaching of maths. Ideas for the classroom, personal experiences and shared thoughts about developing learners' understanding.
- Professional development courses tailored to your needs. Agree the content with us and we do the rest.
- Easter conference, which brings together teachers interested in learning and teaching mathematics, with excellent speakers and workshops and seminars led by experienced facilitators.
- Regular e-newsletters keeping you up to date with developments in the learning and teaching of mathematics.
- Generous discounts on a wide range of publications and software.
- A network of mathematics educators around the United Kingdom to share good practice or ask advice.
- Active campaigning. The ATM campaigns at all levels towards: encouraging increased understanding and enjoyment of mathematics; encouraging increased understanding of how people learn mathematics; encouraging the sharing and evaluation of teaching and learning strategies and practices; promoting the exploration of new ideas and possibilities and initiating and contributing to discussion of and developments in mathematics education at all levels.
- Representation on national bodies helping to formulate policy in mathematics education.
- Software demonstrations by arrangement.

Personal members get the following additional benefits:

- Access to a members only part of the popular ATM website giving you access to sample materials and up to date information.
- Advice on resources, curriculum development and current research relating to mathematics education.
- Optional membership of a working group being inspired by working with other colleagues on a specific project.
- Special rates at the annual conference
- Information about current legislation relating to your job.
- Tax deductible personal subscription, making it even better value

Additional benefits

The ATM is constantly looking to improve the benefits for members. Please visit www.atm.org.uk regularly for new details.

LINK: www.atm.org.uk/join/index.html