From Archimedes to limits, Bob Burn

Part 1. The Spiral and its applications in the 17th century
No limits here.

1. Is there a smallest positive number? or a greatest *negative* number? Argue by contradiction.

2. If $-\varepsilon < A < \varepsilon$, for all positive ε, what may be said about A?
This theorem will be used repeatedly, and will be called "The Vice".

What makes analysis so different from algebra or geometry is its repeated use of inequalities to obtain equalities.

3. [*Archimedean Order*] In two of Archimedes letters, he articulates a principle (also to be found in Euclid V. Def.4) that if two quantities are given, some multiple of the first will exceed the second. This axiom excludes infinitesimals. It allows the comparison of two line segments or two areas or two volumes, but not a ratio of a line segment to an area.

Use this axiom, with 2 (above) to show that if
$-1/n < A < 1/n$, for all positive integers n, then $A = 0$.

Archimedes had neither algebra, nor negative numbers, so he did not express things this way. Without negative numbers, the two inequalities are established by independent arguments. If you find Archimedean Order awkward to use, you may be surprised to realise that you depend on it every time you want to say $(1/n) \to 0$.

4. Can you extend the result of 3 to show that when B and C are constant positive quantities, and $-B/n < A < C/n$ for all positive integers n, then $A = 0$?

5. If U is an unknown quantity, and K is a known quantity, and
$-B/n < U - K < C/n$, for all positive integers n, prove that $U = K$.

This is the structure behind many of Archimedes' determinations of areas and volumes.

6. [*The Spiral*] An example (a rather significant example) of how Archimedes used the idea of 5 is his quadrature of the spiral. He showed that the area within one circuit of the spiral $r = a\theta$ was equal to one third of the area of the circumscribing circle. We examine the part of the spiral from $\theta = 0$ to $\theta = 2\pi$. The radius of the circumscribing circle is $2\pi a$ and the circle is divided into n equiangular sectors. Within each sector, say from $\theta = 2\pi(i - 1)/n$ to $\theta = 2\pi i/n$, we compare that part of the spiral with the largest circular sector *inside* the spiral and the smallest circular sector outside the spiral to get:

$$\tfrac{1}{2}[2\pi a(i-1)/n]^2[2\pi/n] < \text{portion of spiral} < \tfrac{1}{2}[2\pi a i/n]^2[2\pi/n].$$

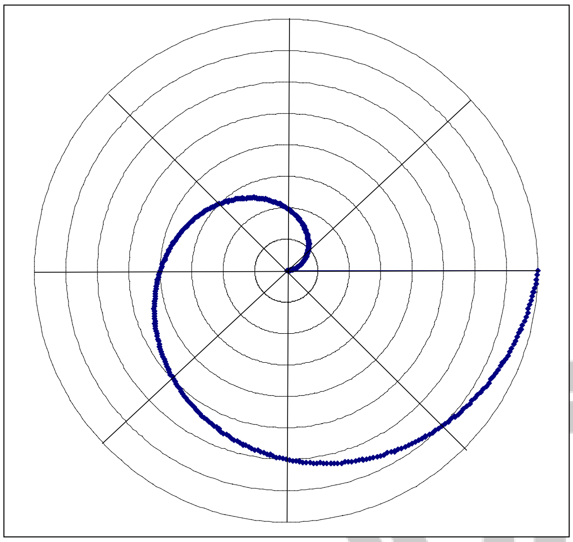

Figure 1. Spiral, circumscribing circle and sectors of angle $2\pi/n$, for $n = 8$.

(The area of a circular sector is $\frac{1}{2}r^2\theta$ where r is the radius of the sector and θ the angle at the centre.) Adding the inscribed sectors for $i = 1, \ldots, n$ we get,

$$\sum_{i=1}^{n} \frac{1}{2}\left(\frac{2\pi a(i-1)}{n}\right)^2\left(\frac{2\pi}{n}\right) = \left(\frac{4\pi^3 a^2}{n^3}\right)\left(1^2 + 2^2 + \ldots + (n-1)^2\right) < \text{area of spiral.}$$

Adding the circumscribed sectors for $i = 1, \ldots, n$ we get,

$$\sum_{i=1}^{n} \frac{1}{2}\left(\frac{2\pi ai}{n}\right)^2\left(\frac{2\pi}{n}\right) = \left(\frac{4\pi^3 a^2}{n^3}\right)\left(1^2 + 2^2 + \ldots + n^2\right) > \text{area of spiral.}$$

Now Archimedes worked out the sum $1^2 + 2^2 + \ldots + n^2 = (n/6)(n + 1)(2n + 1)$ in the middle of his proof. Using this result, the sum of the areas of the inscribed sectors is

$$4\pi^3a^2(1/6)(1 - 1/n)(2 - 1/n),$$

and the sum of the areas of the circumscribed sectors is

$$4\pi^3a^2(1/6)(1 + 1/n)(2 + 1/n).$$

So if S is the area of the spiral

$$4\pi^3a^2(1/6)(1 - 1/n)(2 - 1/n) < S < 4\pi^3a^2(1/6)(1 + 1/n)(2 + 1/n).$$

Now the area of the circumscribed circle is $\pi(2\pi a)^2 = C$, say. So, we set up a vice by getting

$$4\pi^3a^2(1/6)(1-1/n)(2-1/n) - \tfrac{1}{3}C < S - \tfrac{1}{3}C < 4\pi^3a^2(1/6)(1+1/n)(2+1/n) - \tfrac{1}{3}C.$$

However,

$$4\pi^3a^2(1/6)(1 - 1/n)(2 - 1/n) - \tfrac{1}{3}\pi(2\pi a)^2 = C(-3/2n + 1/2n^2)$$

and

$$4\pi^3a^2(1/6)(1 + 1/n)(2 + 1/n) - \tfrac{1}{3}\pi(2\pi a)^2 = C(3/2n + 1/2n^2).$$

So

$$C(-3/2n + 1/2n^2) < S - \tfrac{1}{3}C < C(3/2n + 1/2n^2),$$

and since $1/n^2 \leq 1/n$, we can say that $-2C/n < S - \frac{1}{3}C < 2C/n$.

Now this holds for all positive integer values of n, so using the Archimedean postulate,

$$-\varepsilon < S - \tfrac{1}{3}C < \varepsilon$$

for all positive ε, and it follows that $S = \frac{1}{3}C$.

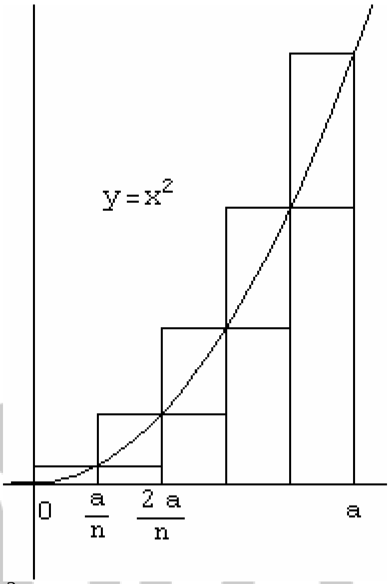

Figure 2. Parabola $y = x^2$, with inscribed and circumscribed rectangular strips

7. Use Archimedes' argument in 6 to obtain Fermat's quadrature of the parabola in 1636: the area bounded by x-axis, $x = a$ and the parabola $y = x^2$ equals $\frac{1}{3}a^3$. This can be found by working with rectangular strips, parallel to the y-axis, of width a/n, starting by calculating the area of the inscribed rectangles (like the inscribed sectors of the spiral), and then the area of the circumscribed rectangles (like the circumscribed sectors of the spiral). See figure 2.

Fermat had been extending Archimedes work on $r = a\theta$ to other spirals before he found that the method could be used for other curves.

8. Use Archimedes' argument in 6 to obtain Eudoxus' result (given by a much harder method in Euclid XII) that the volume of square-based pyramid = $\frac{1}{3}$ base area × height. Work with square prisms parallel to the base of the pyramid and of thickness (height)/n.

9. Use Archimedes' argument in 6 to obtain Eudoxus' result (given by a much harder method in Euclid XII) that the volume of a right circular cone is = $\frac{1}{3}$ base area × height. Work with cylindrical discs parallel to the base and of thickness (height)/n.

10. Use Archimedes' argument in 6 to obtain Fermat's quadrature of the 'higher parabola' in 1636: the area bounded by x-axis, $x = a$ and the curve $y = x^3$ equals $\frac{1}{4}a^4$. This can be found by working with rectangular strips parallel to the y-axis of width a/n. This needs $\sum_{i=1}^{n} i^3 = \left[\frac{1}{2}n(n+1)\right]^2$, which was known to the Arabs.

11. Having found the volume of a cone in 9, one can find that the volume of a paraboloid formed by revolving $y = x^2$ about the y-axis and cut off by the plane $y = R^2$ equals one and a half the volume of the cone with the same base and vertex at O. This was obtained by Archimedes, in his book *On Conoids and Spheroids*, by working with circular discs parallel to the xz-plane and of thickness R^2/n. The method is algebraically similar to the proofs above, though, rather than the sum of the squares, this proof needs $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$.

12. Another result obtained by Fermat in 1636: the area bounded by the spiral $r^2 = a\theta$ and the line $\theta = 0$ is one half the area of the circumscribing circle. This also needs the Σr formula as in 11, above.

Part II. The Quadrature of the Parabola and Geometric Progressions.

1. The Principle of Archimedean Order appears in Euclid Book V, Definition 4

2. Use mathematical induction to show that $2^{n-1} \geq n$ for all positive integers n. Euclid of course did not state an inductive argument, but he certainly understood the idea in this case. Deduce that $(½)^{n-1} \leq 1/n$, for all positive integers n.

3. Making use of the vice obtained in part I from $-1/n < A < 1/n$ for all positive integers n to establish $A = 0$, deduce that $-(½)^{n} < A < (½)^{n}$ for all positive integers n, also gives $A = 0$.