ADDRESSING COMMON ERRORS IN MATHEMATICS THROUGH JOINED-UP MEASURES

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In this paper, the author intends to discuss the topics that are sources of confusion to students. Such sources of confusion are the conventions that need to be emphasized and discussed thoroughly with the students. The common errors will be presented and joined-up measures that can help students assimilate the correct ideas will be suggested.

I. On Performing Operations

The key here is to first know the order of operations to perform. Thus, we have to start with knowing the MDAS rule.

MDAS rule is such that in doing arithmetic operations, the sequence is as follows: Multiplication and Division (MD) primarily and doing whichever comes as you work from left to right, followed by Addition and Subtraction (AS) doing again whichever comes from left to right.

In a diagnostic exam administered to 190 first year college students, the expression $15 \div 3 \cdot 5 - 3 + 4$ was asked to be simplified and only 82 got the right answer. This means that 57% of the involved students do not know how to apply MDAS correctly.

Common mistake of students here is assuming that MDAS means you do multiplication first, then division, then addition and lastly subtraction.
Joined-up measures: We can make students simplify simple expressions like $15 \div 3 \cdot 5 - 3 + 4$.

The following are subtopics that students need to make clear in their minds.

A. Differentiating negative from minus sign

The symbol $-a$ is read as negative a. So that -3 is negative three. It may be interpreted as a lacking, wanting or deficient of 3.
We also have $-a = (-1)a$. This means the symbol -3 may also be written as $(-1)3$, that is, negative one times three.

The symbol $a-b$ is a minus b. Thus, $5-3$ is five minus three and not five negative three.

We also have the rule $-(-a) = a$, that is, the negative of negative a is a. If we write $-(-3)$, this is negative of negative three which is the same as three.
$-2+-3$ is negative two plus negative three.
$2-(-3-7)$ is two minus negative three minus seven.
$2--(-3-4)$ is two minus negative of negative three minus four.

Common error: $5-(-3)=5\cdot 3=15$

Joined-up measures: First, we can give exercises on reading expressions correctly. For example, we can ask them to read the expression $2--(-3-4)$ orally. We can make the students answer $5-(-3)$ which is five minus negative three and not five negative of negative three. We can also ask them to simplify more complex things such as $2--(-3-4)$.

We may show solution with the common error above and ask the students to identify the mistake.

B. Expressing subtraction in terms of addition

The operation of subtraction may be defined in terms of addition. That is, $a-b=a+-b=-b+a$. This means a minus b is the same as a plus negative b. So that $5-3=5+-3$.

Another example is $5-(-3)=5+-(-3)$ which is five minus negative three is same as five plus negative of negative three. From A above, negative of negative three is three. Thus we have $5-(-3)=5+-(-3)=5+3=8$. Knowing this relationship between subtraction and addition, students can just focus on addition of signed numbers.

Common error: $-3-6=3$

Joined-up measures: Give problems like $-3-6$ and ask the students to make use of the rules on adding signed numbers.

C. Acknowledging the signs of grouping

The signs of grouping give ideas as to the order of operations. Any expression inside a pair of signs of grouping may be treated as one number. Also, the juxtaposition for multiplication is used for expressions involving symbols other than numerals. For example, $2x$ means 2 times x.

The expression $2x+y$ means x is multiplied by 2 and y is added to the product. Note that this also makes use of the MDAS rule. If what we want is for $x+y$ to be multiplied by 2, we write $2(x+y)$.

Now the juxtaposition does not apply when we have pure numbers. Such that the expression 23 does not mean 2 times 3, it means twenty three. Also, $5-3$ does not mean 5 times -3, it is five minus three.

Joined-up measures: make them solve and discuss $4 - 2\cdot[3 - (2+12\div 4)+1] - 5\cdot 2$

We can also have $[x-3(x+2)]$ which is the same with $x-3(x+2)$ but is not the same as $(x-3)(x+2)$. It might also work if we show them a wrong solution where we have $[x-3(x+2)]=(x-3)(x+2)=x^2-x-6$ and ask them to determine whether the solution is right or wrong. With this, we have to ask them to justify their answer.

D. Decimal numbers

On decimal numbers, we can start with discussion on equivalences of the following: $1.756 = 1 + .756$ and $-2.143 = -2 + -.143 = -2 - .143$

Common error: $1.756 - 2.143 = 1 - 2 + .756 - .143 = -1.613$

Joined-up measures: Make students solve $1.756 - 2.143$ directly or show the solution in the common error above that will have -1.613 as answer. Ask them what is wrong with the solution.

II. On fractions

We need to be clear on some ideas that we usually assume the students know.

First and foremost, for secondary and tertiary levels, we have to let students know that the notation for mixed fractions cannot be used for algebraic expressions. That is, the conventionally accepted notation for mixed fractions cannot be carried over to algebraic expressions or expressions involving letters representing numbers. The misconception here is otherwise.

We then have, by convention, $5\frac{3}{4}$ can stand for $5 + \frac{3}{4}$, but the notation $x\frac{y}{z}$ cannot stand for $x + \frac{y}{z}$.

In a separate survey using a short questionnaire conducted to 120 students, it has been revealed that 65 students wrongly believe that what lies behind the fact that $5\frac{3}{4} = 5 + \frac{3}{4}$ can be applied to algebraic expressions, thus making $x\frac{y}{z} = x + \frac{y}{z}$ also acceptable. Out of the 55 who do not believe that this is so, it is worth mentioning that 48 actually do not know that $5\frac{3}{4} = 5 + \frac{3}{4}$. This means that only 7 students have the correct idea on the fact that lies here.

Joined-up measures: It would help if we actually give examples like the following:

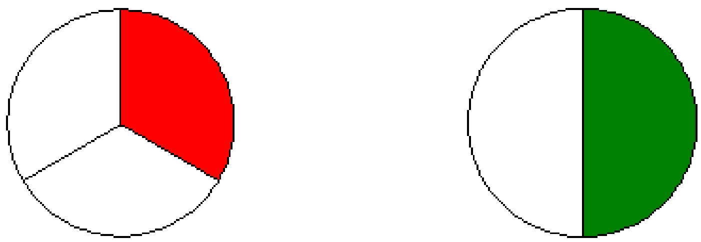
$$5\left(\frac{3}{4}\right) = 5 \cdot \frac{3}{4};\ 2\frac{x}{y} = 2\left(\frac{x}{y}\right) = 2 \cdot \frac{x}{y};\ w\frac{v+3}{z} = w\left(\frac{v+3}{z}\right) = w \cdot \frac{v+3}{z}.$$

Realization of this truth will help a lot in the correct manipulation of algebraic expressions that the students will do. Note that mistakes such as having $x + \frac{y}{x} = y$ as a result of canceling the x may come from the wrong belief that $x + \frac{y}{x} = x\frac{y}{x}$ and the fact that $x\frac{y}{x} = y$.

Other difficulties related to fractions are addressed in the following subtopics.

A. Comparing fractions

When we speak of fractions we have to consider the fact that in dealing with two or more fractions, we are actually assuming that these fractions refer to exactly the same idea of what one whole is. That is, we are dealing with the same "scale" or the same "unit measure". This means that when we compare $\frac{1}{3}$ and $\frac{1}{2}$, we have the assumption that both have the same reference to what one whole is. That is, we refer to the same idea of wholeness for the two fractions. Thus, we may think of actually comparing the measure of the shaded parts in the following circles of exactly the same size:



one-third (1/3) one-half (1/2)

Figure 1

Looking at the figures above, we can say that one-half is obviously bigger than one-third.

We can also discuss equal fractions using possible divisions of the circle that will preserve the shaded part. See the figures below.

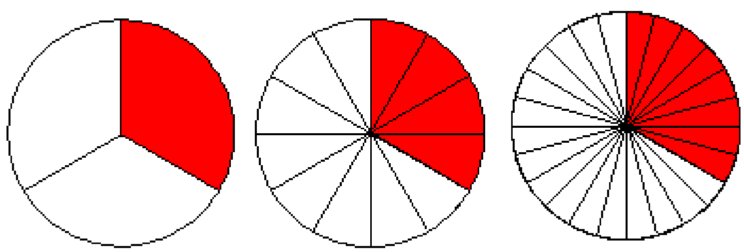


Figure 2

Note that based on what we have in the figures above, the fractions $\frac{1}{3}$, $\frac{4}{12}$ and $\frac{8}{24}$ are equal to each other. That is $\frac{1}{3}=\frac{4}{12}=\frac{8}{24}$.

Joined-up measures: We can ask students to give equivalent fractions to $\frac{1}{2}$.

B. Addition of fractions

We may also use the method above in discussing the sum of fractions with students. For example, if we add $\frac{1}{3}$ and $\frac{1}{2}$, we will get the following figure:

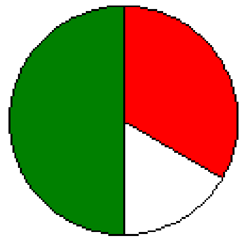


Figure 3

The shaded part (both green and red) will form the sum of the fractions $\frac{1}{3}$ and $\frac{1}{2}$. With this, we can take equal partitions of the circle, to show the need for finding the common denominator when adding fractions.

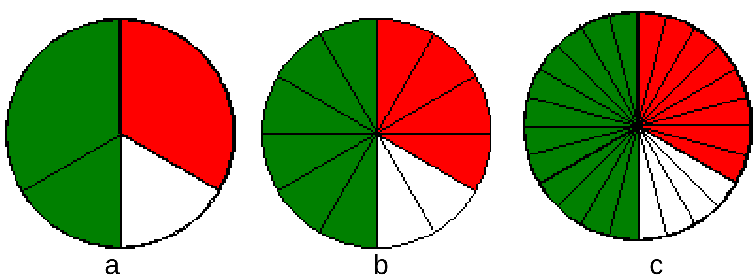


Figure 4

From the figures above, we can see that the division by 12 equal parts and by 24 equal parts can give us the measure of the shaded (green and red) part. On the other hand, the division by 3 equal parts will have part of the green region and the white region together in one partition. With this we can say that division by three equal parts will not enable us to count the number of parts shaded in relation to the whole. With figure 4b, we can get that the shaded part is $\frac{10}{12}$. From figure 4c, it is $\frac{20}{24}$. This would mean that $\frac{1}{3}+\frac{1}{2}=\frac{10}{12}$ or $\frac{1}{3}+\frac{1}{2}=\frac{20}{24}$. We can also have the division below.

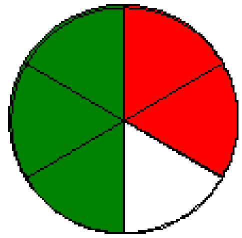


Figure 5

With the figure above, we can say that $\frac{1}{3}+\frac{1}{2}=\frac{5}{6}$. Thus, $\frac{5}{6}=\frac{10}{12}=\frac{20}{24}$.

Joined-up measures: ask students to add $\frac{1}{3}$ and $\frac{1}{4}$.

C. Simplifying fractions

In simplifying fractions we use mainly the rule $\frac{ac}{bc}=\frac{a}{b}$. This can be further explored with the aid of the result above. We saw that $\frac{5}{6}=\frac{10}{12}=\frac{20}{24}$. Note that $\frac{10}{12}=\frac{5(2)}{6(2)}=\frac{5}{6}$ and $\frac{20}{24}=\frac{5(4)}{6(4)}=\frac{5}{6}$.

Common error: $\frac{9}{4}$ is reduced to $\frac{3}{2}$.

Joined-up measures: We can ask students to simplify $\frac{8}{12}, \frac{9}{15}, \frac{5x}{6x}, \frac{3(x+2)}{4(x+2)}$.

III. On proving

A statement may not be used to directly prove itself. The misconception is exactly the opposite.

An instrument conducted to 120 students resulted to the discovery that in proving $(x-y)(x+y)=x^2-y^2$, 58 students will approve of starting the proof with the statement $(x-y)(x+y)=x^2-y^2$ itself.

Students need to be told that a statement cannot be used to directly prove itself. The statement may be used in getting ideas on how it can be proven but it cannot be used in the proof itself. Particular example is on proving identities involving circular functions. The proof may be accomplished by (a) transforming the left-hand side of the equation into the exact form of the expression on the right-hand side, (b) transforming the right into the exact form of the left, or (c) transforming each side separately into the same form and making use of transitivity. In any method, it must be emphasized that we cannot start with the given equation that needs to be proven in the formal presentation of the proof.

For example, if we are to prove that $\tan\theta\sin\theta \equiv \sec\theta - \cos\theta$, we can have any of the following proofs:

1) $\tan\theta\sin\theta = \dfrac{\sin\theta}{\cos\theta}\sin\theta = \dfrac{\sin^2\theta}{\cos\theta} = \dfrac{1-\cos^2\theta}{\cos\theta} = \dfrac{1}{\cos\theta} - \cos\theta = \sec\theta - \cos\theta$

2) $\sec\theta - \cos\theta = \dfrac{1}{\cos\theta} - \cos\theta = \dfrac{1-\cos^2\theta}{\cos\theta} = \dfrac{\sin^2\theta}{\cos\theta} = \dfrac{\sin\theta}{\cos\theta}\sin\theta = \tan\theta\sin\theta$

3) We can use transitivity with

$\tan\theta\sin\theta = \dfrac{\sin\theta}{\cos\theta}\sin\theta = \dfrac{\sin^2\theta}{\cos\theta}$ and

$\sec\theta - \cos\theta = \dfrac{1}{\cos\theta} - \cos\theta = \dfrac{1-\cos^2\theta}{\cos\theta} = \dfrac{\sin^2\theta}{\cos\theta}$

4) We may also have the following formal proof

$$\frac{\sin^2\theta}{\cos\theta} = \frac{\sin^2\theta}{\cos\theta}$$
$$\frac{\sin^2\theta}{\cos\theta} = \frac{1-\cos^2\theta}{\cos\theta}$$
$$\frac{\sin\theta}{\cos\theta}\sin\theta = \frac{1}{\cos\theta} - \cos\theta$$
$$\tan\theta\sin\theta = \sec\theta - \cos\theta$$

What needs to be emphasized is that the proof below is not acceptable because it starts with the very statement that needs to be proven.

$$\tan\theta\sin\theta = \sec\theta - \cos\theta$$
$$\frac{\sin\theta}{\cos\theta}\sin\theta = \frac{1}{\cos\theta} - \cos\theta$$
$$\frac{\sin^2\theta}{\cos\theta} = \frac{1-\cos^2\theta}{\cos\theta}$$
$$\frac{\sin^2\theta}{\cos\theta} = \frac{\sin^2\theta}{\cos\theta}$$

Not being able to accept this truth might lead to wrong reasoning brought about by non-determination of the difference between premises and conclusions. The very good contribution of mathematics in the learning process, that is developing logical thinking in the students, may not be achieved.

IV. References

Vance, Elbridge, *College Algebra and Trigonometry*, Addison-Wesley Publishing Co., Inc., 1973.

Leithold, Louis, *College Algebra and Trigonometry*, Addison-Wesley Publishing Co., Inc., 1991.

T. Endozo, E. Rodriguez, N. Torre, G. Tumbocon, and M.V.P. Quilinguin (ed.), *Basic Math*, Inkwell Publishing Company, Inc., 2005.